HEGEL ON BEING

VOLUME 2

HEGEL ON BEING

QUANTITY AND MEASURE IN HEGEL'S SCIENCE OF *LOGIC*

VOLUME 2

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PREFACE TO VOLUME 2

In volume 1 of *Hegel on Being* I first explained the purpose and method of Hegel's logic. Its distinctive feature, in my view, is that it is (or is intended to be) the free, presuppositionless derivation of categories that belong to both thought and being. It is thus both a logic and an ontology at the same time. Since this logic is to have no systematic presuppositions, it must begin with a category that is utterly indeterminate, namely pure being. It then proceeds by considering what further categories, if any, are to be derived from this indeterminate starting point. In Part Two of volume 1 we saw that a whole series of categories are derived from being, including *something*, *finitude*, *infinity* and the *one* (*Eins*). These categories of "quality" constitute, for Hegel, the most basic categories of thought, but also the most basic ways of being. We usually just take it for granted that there are "things" in the world and that they will at some point cease to be. Hegel, however, shows why there must be "things", and why they must cease to be, by showing that being itself makes it necessary logically that there be finite things.

At the end of his account of quality, Hegel argues that being also makes quantity necessary. It does so via the quality of being *one*. The one, Hegel claims, proves to be one among many but thereby to continue beyond itself, and in so doing it gives rise to being that consists specifically in the continuity of discrete units. This being is quantity. For Hegel, therefore, quantity is not just a contingent feature of things, but they must have quantity because – besides being finite – they are *ones*, and being one logically entails being many-in-being-one and thus being quantitative.

Note that with the logical transition from quality to quantity, what it is to be qualitative does not itself change. It still comprises all the categories considered in volume 1. Quality gives rise to quantity, however, by proving no longer just

to be qualitative. In quantity, therefore, there is – at least initially – no longer any quality as such. Quantity is constituted by the qualitative categories of the one, repulsion and attraction (in subtly altered forms), but quality as such has – for now – disappeared.

In volume 1, however, I considered only the birth of quantity from quality and the fact that quantity is by its nature divisible (without already being divided into parts). I then discussed Hegel's analysis and critique of Kant's second antinomy, which, for Hegel, is grounded in the nature of quantity. The task of volume 2 is now to examine quantity in more detail and to discover what further categories are made necessary by it.

NOTE ON REFERENCES AND ABBREVIATIONS

References to works by Kant, Hegel, Schelling and Frege are provided in the main body of the text (unless they are particularly long or would otherwise clutter the text, in which case they are provided in the notes). Other references are provided in the notes. Where a single-volume English translation corresponds to a single-volume German original, I have used one abbreviation for both texts (e.g. EL, LL, PS). In the case of SL, however, a single-volume English translation corresponds to three different German texts, and in the case of LHP a three-volume English translation corresponds to a four-volume original, so I have used different abbreviations for the English and German texts. References to Hegel's *Encyclopaedia* are thus given in the form: EL 45 / 67 [§ 19] (or EN 28 / 41 [§ 253]), whereas references to the *Science of Logic* are given in the form: SL 59 / LS 71 (or SL 337 / LW 3). In all cases the English translation is cited first, followed by the German text. Where necessary, line numbers are provided after the page number.

In references to the *Encyclopaedia*, "§ 19" refers to the paragraph number, "R" to the Remark, and "A" to the Addition. (The paragraphs and remarks are by Hegel, but the additions were compiled by later editors on the basis of student transcripts of Hegel's lectures.) References to Kant's *Critique of Pure Reason* are given in the usual form (citing the A and / or B editions): CPR A 19 / B 33.

Note that in all cases in which several page numbers are cited in one reference, I have given them in numerical order, even though this may not match the order of the passages referred to. Note, too, that at certain points in the text I have identified sub-divisions of the *Science of Logic* by using Hegel's system of numbers and letters. So, for example, 1.1.1.C.1 refers to Book 1

(Doctrine of Being), Section 1 (Quality), chapter 1 (Being), sub-section C (Becoming), sub-sub-section 1 (Unity of Being and Nothing) (see SL 59 / LS 72). Cross-references to other parts of *Hegel on Being* are given in the form: 1: 71-2 (for volume 1) or 2: 330-1 (for volume 2).

Finally, note that I have occasionally altered translations, but I have not indicated in each case that such an alteration has been made. Where this makes the relevant passage hard to find in the English translation, I have cited the unaltered English version in a note or provided line numbers. In the case of di Giovanni's translation of Hegel's *Science of Logic* – which is the principal one used in this study – I have sometimes altered this translation myself and sometimes done so by drawing on Miller's translation (SLM). Where no published English translation of a French, German or Latin text is cited, the translation is my own.

ABBREVIATIONS

KANT

CPR Kant, Immanuel, *Critique of Pure Reason*, trans. P. Guyer and A.W. Wood (Cambridge: Cambridge University Press, 1997).

CPR Kant, Immanuel, *Kritik der reinen Vernunft*, ed. R. Schmidt (Hamburg: Felix Meiner, 1990).

HEGEL

EL Hegel, G.W.F., *The Encyclopaedia Logic (with the Zusätze). Part I of the Encyclopaedia of Philosophical Sciences with the Zusätze*, trans. T.F. Geraets, W.A. Suchting, and H.S. Harris (Indianapolis: Hackett Publishing, 1991).

EL Hegel, G.W.F., *Enzyklopädie der philosophischen Wissenschaften im Grundrisse (1830), Erster Teil: Die Wissenschaft der Logik. Mit den mündlichen Zusätzen*, ed. E. Moldenhauer and K.M. Michel, *Werke in zwanzig Bänden*, vol. 8 (Frankfurt am Main: Suhrkamp, 1970).

EN Hegel, G.W.F., *Philosophy of Nature. Being Part Two of the Encyclopaedia of the Philosophical Sciences (1830)*, trans. A.V. Miller (Oxford: Clarendon Press, 1970).

EN Hegel, G.W.F., *Enzyklopädie der philosophischen Wissenschaften im Grundrisse (1830), Zweiter Teil: Die Naturphilosophie. Mit den mündlichen Zusätzen*, ed. E. Moldenhauer and K.M. Michel, *Werke in zwanzig Bänden*, vol. 9 (Frankfurt am Main: Suhrkamp, 1970).

EPM Hegel, G.W.F., *Philosophy of Mind (1830) with the Zusätze*, trans. W. Wallace and A.V. Miller, revised with introduction and commentary by M.J. Inwood (Oxford: Clarendon Press, 2007).

EPM Hegel, G.W.F., *Enzyklopädie der philosophischen Wissenschaften im Grundrisse (1830), Dritter Teil: Die Philosophie des Geistes. Mit den mündlichen Zusätzen*, ed. E. Moldenhauer and K.M. Michel, *Werke in zwanzig Bänden*, vol. 10 (Frankfurt am Main: Suhrkamp, 1970).

LB Hegel, G.W.F., *Wissenschaft der Logik. Zweiter Band: Die subjektive Logik oder die Lehre vom Begriff (1816)*, ed. H.-J. Gawoll (Hamburg: Felix Meiner, 2003).

LHP Hegel, G.W.F., *Lectures on the History of Philosophy, 1825-6*, ed. R.F. Brown, trans. R.F. Brown and J.M. Stewart, 3 vols. (Oxford: Clarendon Press, 2006-9).

LS Hegel, G.W.F., *Wissenschaft der Logik. Erster Teil: Die objektive Logik. Erster Band: Die Lehre vom Sein (1832)*, ed. H.-J. Gawoll (Hamburg: Felix Meiner, 2008).

LSGW Hegel, G.W.F., *Wissenschaft der Logik. Erster Teil: Die objektive Logik. Erster Band: Die Lehre vom Sein (1832)*, ed. F. Hogemann and W. Jaeschke, *Gesammelte Werke*, Band 21 (Hamburg: Felix Meiner, 1985).

LW Hegel, G.W.F., *Wissenschaft der Logik. Erster Band: Die objektive Logik. Zweites Buch: Die Lehre vom Wesen (1813)*, ed. H.-J. Gawoll (Hamburg: Felix Meiner, 1999).

NP Hegel, G.W.F., *Naturphilosophie. Die Vorlesung von 1819/20*, ed. M. Gies (Naples: Bibliopolis, 1982).

PN Hegel, G.W.F., *Philosophy of Nature*, ed. and trans. M.J. Petry, 3 vols. (London: Allen and Unwin, 1970).

PS Hegel, G.W.F., *Phenomenology of Spirit*, trans. A.V. Miller (Oxford: Oxford University Press, 1977).

PS Hegel, G.W.F., *Phänomenologie des Geistes*, ed. H.-F. Wessels and H. Clairmont (Hamburg: Felix Meiner, 1988).

SL Hegel, G.W.F., *The Science of Logic*, trans. and ed. G. di Giovanni (Cambridge: Cambridge University Press, 2010).

SLM Hegel, G.W.F., *Science of Logic*, trans. A.V. Miller (Amherst: Humanity Books, 1999).

VGP Hegel, G.W.F., *Vorlesungen über die Geschichte der Philosophie*, ed. P. Garniron and W. Jaeschke, 4 vols. (Frankfurt am Main: Felix Meiner, 1986-96).

VGPW Hegel, G.W.F., *Vorlesungen über die Geschichte der Philosophie*, ed. E. Moldenhauer and K.M. Michel, 3 vols., *Werke in zwanzig Bänden*, vols. 18, 19, 20 (Frankfurt am Main: Suhrkamp, 1971).

WL Hegel, G.W.F., *Wissenschaft der Logik*, ed. E. Moldenhauer and K.M. Michel, 2 vols. *Werke in zwanzig Bänden*, vols. 5, 6 (Frankfurt am Main: Suhrkamp, 1969).

WLS Hegel, G.W.F., *Wissenschaft der Logik. Erster Band: Die objektive Logik. Erstes Buch: Das Sein (1812)*, ed. H.-J. Gawoll (Hamburg: Felix Meiner, 1999).

SCHELLING

AS Schelling, F.W.J. von, *Ausgewählte Schriften*, ed. M. Frank, 6 vols. (Frankfurt am Main: Suhrkamp, 1985).

FREGE

ASB Frege, Gottlob, "Ausführungen über Sinn und Bedeutung", in Frege, *Schriften zur Logik und Sprachphilosophie*, ed. G. Gabriel (Hamburg: Felix Meiner, 2001), pp. 25-34. ["Comments on *Sinn* and *Bedeutung*": translation in *The Frege Reader*, pp. 172-80.]

B Frege, Gottlob, *Briefwechsel mit D. Hilbert, E. Husserl, B. Russell*, ed. G. Gabriel, F. Kambartel and C. Thiel (Hamburg: Felix Meiner, 1980).

BS Frege, Gottlob, *Begriffsschrift und andere Aufsätze*, ed. I. Angelelli (Hildesheim: Olms, 2007). [*Concept-script*: partial translation in *The Frege Reader*, pp. 47-78 under the title "*Begriffsschrift*".]

FB Frege, Gottlob, "Funktion und Begriff", in Frege, *Funktion, Begriff, Bedeutung. Fünf logische Studien*, ed. G. Patzig (Göttingen: Vandenhoeck & Ruprecht, 2008), pp. 1-22. ["Function and Concept": translation in *The Frege Reader*, pp. 130-48.]

FR *The Frege Reader*, ed. M. Beaney (Oxford: Blackwell, 1997).

G Frege, Gottlob, "Der Gedanke. Eine logische Untersuchung", in Frege, *Logische Untersuchungen*, ed. G. Patzig (Göttingen: Vandenhoeck & Ruprecht, 1993), pp. 30-53. ["Thought": translation in *The Frege Reader*, pp. 325-45.]

GA Frege, Gottlob, *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, ed. J. Schulte (Stuttgart: Reclam, 1987). [*The Foundations of Arithmetic*: partial translation in *The Frege Reader*, pp. 84-129.]

GGA Frege, Gottlob, *Grundgesetze der Arithmetik*, vol. 1 (Jena: Hermann Pohle, 1893). [*Basic Laws of Arithmetic* 1: partial translation in *The Frege Reader*, pp. 194-223 under the title "*Grundgesetze der Arithmetik*, volume 1".]

L Frege, Gottlob, "Logik", in Frege, *Schriften zur Logik und Sprachphilosophie*, ed. G. Gabriel (Hamburg: Felix Meiner, 2001), pp. 35-73. ["Logic": partial translation in *The Frege Reader*, pp. 227-50.]

LA Frege, Gottlob, "Logische Allgemeinheit", in Frege, *Schriften zur Logik und Sprachphilosophie*, ed. G. Gabriel (Hamburg: Felix Meiner, 2001), pp. 166-71.

LM Frege, Gottlob, "Logik in der Mathematik", in Frege, *Schriften zur Logik und Sprachphilosophie*, ed. G. Gabriel (Hamburg: Felix Meiner, 2001), pp. 92-165. ["Logic in Mathematics": partial translation in *The Frege Reader*, pp. 308-18.]

OFG Frege, Gottlob, "On the Foundations of Geometry", in Frege, *On the Foundations of Geometry and Formal Theories of Arithmetic*, ed. E.-H. W. Kluge (New Haven: Yale University Press, 1971), pp. 22-37, 49-112.

ÜBG Frege, Gottlob, "Über Begriff und Gegenstand", in Frege, *Funktion, Begriff, Bedeutung. Fünf logische Studien*, ed. G. Patzig (Göttingen: Vandenhoeck & Ruprecht, 2008), pp. 47-60. ["On Concept and Object": translation in *The Frege Reader*, pp. 181-93.]

ÜGG Frege, Gottlob, "Über die Grundlagen der Geometrie", *Jahresbericht der deutschen Mathematiker-Vereinigung* 12 (1903): 319-24, 368-75.

ÜSB Frege, Gottlob, "Über Sinn und Bedeutung", in Frege, *Funktion, Begriff, Bedeutung. Fünf logische Studien*, ed. G. Patzig (Göttingen: Vandenhoeck & Ruprecht, 2008), pp. 23-46. ["On *Sinn* and *Bedeutung*": translation in *The Frege Reader*, pp. 151-71.]

PART ONE

Quantum, Number and the Quantitative Ratio

CHAPTER ONE

Quantum and Number

CONTINUOUS AND DISCRETE MAGNITUDE

According to speculative logic, quantity is the unity of continuity and discreteness, but, as we saw in volume 1, it is first of all "in the *form* of one of them, of *continuity*" (SL 154-5 / LS 195).[1] Quantity is initially simple, self-relating being or "self-identical immediacy"; yet it also contains a multiplicity of discrete ones; it is thus self-relating being that *continues through* the many ones within it. Accordingly, quantity is initially – as the "*immediate* unity" of continuity and discreteness – *continuous* being above all (SL 165-6 / LS 209). As continuous, quantity cannot be "composed" of units that are wholly discrete. It is, however, divisible *into* discrete units, and so, as Hegel puts it, is the "*real possibility* of the one" (SL 155 / LS 196).

Yet quantity is not in fact just the unity of continuity and discreteness, but it also contains their difference. Indeed, the difference between them is built into their unity: for quantity unites continuity *and* discreteness. In truth, therefore, quantity is not merely the "*immediate* unity", but what Hegel calls the "concrete unity", of continuity and discreteness – "the unity of *distinct* moments" (SL 165-6 / LS 209-10).

If, however, continuity and discreteness are to be *explicitly* distinct moments of quantity, the former cannot simply contain the latter (as is initially the case), but must also stand in contrast to it. That in turn means that quantity cannot just be continuous-*in*-being-discrete, but must also be continuous-*rather-than*-discrete. Such overtly one-sided quantity Hegel calls "*continuous magnitude*" (*kontinuierliche Größe*). Equally, discreteness cannot simply be a moment absorbed in the continuity of quantity. It, too, must stand in contrast to continuity and so constitute quantity governed by discreteness *rather than* continuity. Such quantity Hegel names "*discrete magnitude*" (*diskrete Größe*). It turns out, therefore, that quantity must generate two different "kinds" or

"species" (*Arten*) of itself (SL 167 / LS 211). Since each is a form of *quantity*, it unites both continuity and discreteness within itself; yet each differs from the other because a different moment of quantity is "posited", and so to the fore, in it.[2] As we shall see when we reach the idea of the "degree" (*Grad*), not all quantity has to belong to one or the other of these two kinds.[3] The difference between the two is nonetheless inherent in quantity itself.

In his account of determinate being (*Dasein*), Hegel argues that reality and negation are both forms of quality. Reality, however, is quality with an affirmative, rather than a negative, "accent" (SL 85 / LS 105). Continuous and discrete magnitude may also be said to be distinguished by their different "accents". This similarity between the two forms of quantity and of quality is not accidental but reflects a deeper parallel between them: for continuous and discrete magnitude are forms of quantity that is no longer simple immediacy or being (*Sein*), but explicitly *determinate* being (*Dasein*) – being that contains a determinate difference explicitly within itself.[4]

The difference between the two forms of quantity and those of quality should not, however, be overlooked. Reality and negation are *qualitatively* different from one another, so each is – explicitly at least – immediately itself. This is why each is "concealed" in the other: each is hidden in and by the immediacy of the other. Continuous and discrete magnitude, by contrast, are forms of *quantity* and so are not qualitatively different. There is, indeed, a difference between them, but each is "the whole" (*das Ganze*) that contains both continuity and discreteness explicitly within itself. As Hegel puts it in the *Encyclopaedia Logic*, each is "*the same whole* [. . .] posited first under one of its determinations, and then under the other" (EL 160 / 212-13 [§ 100 R]).

Continuous magnitude, therefore, is the "unity of the discrete", but it is nonetheless marked by *continuity* above all (SL 166 / LS 210). It is the kind of quantity that is exhibited by the continuous extension of space and continuous succession of time. Space and time can be divided into "heres" and "nows", since each contains the moment of discreteness or the "principle of the one"; yet, *pace* Zeno, each is still continuous (see 1: 365). Each is thus an instance of quantity that is "being-outside-one-another [*Außereinandersein*] continuing itself without negation as an internally self-same connectedness" (SL 166-7 / LS 210-11).

Discrete magnitude, by contrast, is marked by the separateness of the unit or one. It is explicitly distinct from continuous magnitude and so lacks the overt continuity that characterizes the latter. It does not, therefore, extend smoothly and "without negation"; on the contrary, "discrete magnitude is this outside-one-another [of the ones] as *not* continuous, as interrupted" (SL 166 / LS 210, emphasis added). Such magnitude is thus discontinuous, broken up into discrete ones, and takes the form of an "*aggregate* [*Menge*] of ones". We encounter this kind of quantity, not in the continuous extension of space, but when we are prompted explicitly to ask *how many* units there are.[5] Discrete magnitude is

thus a different kind of quantity from continuous magnitude; one might even say that it is a different "quality" of quantity (provided one bears in mind that it does not differ qualitatively from its counterpart in the way reality differs from negation). Continuous magnitude is precisely *continuous* and uninterrupted and so forms an explicit unity of ones. Discrete magnitude, by contrast, is discontinuous and takes the form of an aggregate of separate ones.

Yet this aggregate is not just a plurality of atoms in a void, atoms that repel one another and are purely for themselves; that plurality belongs to the sphere of quality, not quantity. Discrete magnitude, as a kind of quantity, is an aggregate of ones that form a *continuity* in their discreteness: "because discrete magnitude is quantity, its discreteness is itself continuous" (SL 166 / LS 210). This is due to the fact that the ones are explicitly "the same as one another", the same discrete one. This sameness or "homogeneity" significantly qualifies their discreteness, for it means that they are not merely separate and discrete, but constitute continuous self-same being: the discrete ones together form a *unity*. Such unity is not projected by us onto the ones, but belongs to the ones themselves: it is the unity they form through the continuing *of* their discreteness. The unity still differs from that found in continuous magnitude, because it is constituted by explicitly *discontinuous*, discrete ones. Yet it means that the latter are not just isolated ones, but "the *many of a unity*" (*das* Viele einer Einheit): discrete magnitude is a continuous, *unified* aggregate, rather than a mere aggregate, of discrete ones.[6] In that respect, even discrete magnitude, in Hegel's view, is not simply "composed" of wholly discrete units; the ones in such magnitude are discrete *moments*, rather than quite separate *parts*, of it.[7]

THE LIMITING OF QUANTITY: THE QUANTUM

At the start of 1.2.1.C Hegel again outlines the main features of discrete magnitude. First, he notes, such magnitude has "the one for its principle": what distinguishes it from continuous magnitude is the explicit presence of the discrete, discontinuous one (SL 167 / LS 211). Second, it is, or contains, a "plurality" (*Vielheit*) of such ones. Third, however, it is "essentially constant [*stetig*]", since each one is the *same* discrete one.[8] In discrete magnitude, the ones are thus not just discontinuous, but also form a continuity and unity – "self-continuing as such in the discreteness of the ones". Discrete magnitude is *continuous* after all, thanks to the constancy of the discreteness within it.[9]

Hegel goes on to argue, however, that since the many ones it contains constitute a unity – one continuous being – discrete magnitude must be more than a distinct *kind* of quantity. It must also take the form of a single, unified quantity: "*a* magnitude" or "*one* magnitude" (eine *Größe*) (SL 167 / LS 211). The reason why this should be is as follows. The unity constituted by discrete magnitude is, as a unity, continuous; yet it is what *discrete* magnitude proves to

be – it is an aspect *of* the latter – and so must differ from the unity found in continuous magnitude. It differs from the latter by being a unity that is itself discrete. Discrete magnitude does, indeed, form a unity; but, in contrast to continuous magnitude (which extends without interruption), it forms an explicitly *discrete unity*. As discrete, this unity must be *something* (*Etwas*) with an identity of its own, indeed something that is a discrete *one* (*Eins*); "it is [thus] posited as *one* magnitude, and the determinacy of it is the one" (SL 167 / LS 211-12).[10]

Hegel adds that this discrete unity must be an "*exclusive* one" – a one that shuts out what it is not. It is not immediately clear what justifies this additional claim, since the one in the sphere of quantity is supposed to have lost its negative, exclusive "edge". The justification becomes apparent, however, in the remainder of the paragraph: the unity formed by discrete magnitude must be exclusive because, in differing from continuous magnitude, discrete magnitude is in fact the negation and *limit* of the latter. "Discrete magnitude as such", Hegel writes, "is in its immediacy not supposed to be limited; but as distinguished from continuous magnitude it is a determinate being and a something, whose determinacy is the one and, as in a determinate being, is also first negation [*erste Negation*] and limit" (SL 167 / LS 212). Taken abstractly by itself, discrete magnitude should have no limit, but simply be the continuity formed by overtly *discrete* ones. Yet such magnitude does not occur just by itself, but forms an inseparable pair with continuous magnitude. Moreover, it differs from the latter by being its direct *negation*: it is explicitly *dis*continuous, rather than continuous. As such, discrete magnitude marks the point at which continuous magnitude extends no further, but comes to an end. Discrete magnitude thereby sets a *limit* to its counterpart, and the discrete unity it forms is in turn the negation and limit of the latter. This discrete unity is thus not just a simple, "edgeless" one (*Eins*), but an "*exclusive* one", as in the sphere of quality.

Now in the sphere of quality itself the fact that one something limits an other means that the limit actually belongs to both and so turns each into the same limited something (see SL 99 / LS 123-4). This is partly true here in the sphere of quantity, too. As McTaggart puts it, "Discrete Magnitude, then, shares its Limit with the Continuous Magnitude outside it", and, as Winfield adds, "the way in which continuous and discrete magnitude limit one another renders them the same".[11] Yet this is only part of the quantitative story, for the difference between the two kinds of magnitude is not completely effaced here; or rather it is both effaced and not effaced at the same time. Initially, indeed, that difference is actually reinforced.

Since discrete magnitude sets a limit to continuous magnitude, rather than vice versa, the limit belongs first and foremost to the former, *not* to the latter. The latter is, precisely, continuous and without limits of its own, but it is limited and brought to a stop *by* the former. The limit is thus a feature of discrete magnitude in particular: it belongs to such magnitude because this magnitude is

discontinuous and so brings continuous magnitude to an end. Discrete magnitude does not stand alone but is bound to and differs from continuous magnitude; it differs from its counterpart, however, by containing and setting a limit while its counterpart does not.

This, I think, explains why Hegel focuses initially on the connection between the limit and the discrete unity to which it belongs, not on the relation between the limit and continuous magnitude. At the start of the second paragraph in 1.2.1.C Hegel remarks that "this limit" "is related to the unity [*Einheit*] and is the negation *in it* [*an derselben*]" (SL 167 / LS 212). The unity in question is not that of continuous magnitude, but is that formed by discrete magnitude itself and referred to in the first paragraph. Hegel's point here, therefore, is that discrete magnitude must a form a unity that not only is a something and a one, but also *contains a limit within itself*. Logically, this limit emerges because discrete magnitude – as explicitly discrete and *dis*continuous – differs from and limits continuous magnitude; but it thereby belongs to *discrete* magnitude in particular and the discrete unity that the latter must form.

Since the limit belongs only to discrete magnitude, not to continuous magnitude, it must initially be a limit in – and also *on* – discrete magnitude itself. This limit resides in the discontinuity of the latter and so is the negation of continuity: "the being which is here limited is essentially as continuity" (SL 167 / LS 212). Such continuity, however, must be, for the moment at least, the continuity of *discrete* magnitude. The latter is continuous insofar as the discrete ones it contains are all the same; yet, as we have seen, these ones form a unity that is itself discrete, discontinuous and *a limit*; they thus set a limit to the continuity of discrete magnitude itself and bring the latter to an end.

Yet discrete magnitude is also *continuous* and, as such, necessarily extends beyond any discrete one that it contains. This means that it must extend beyond the discrete unity that it itself forms – "beyond the limit and this one" that it *is* and to which it is thus "indifferent" (SL 167 / LS 212). That means in turn that the ones contained *within* such a limited unity are necessarily marked off from a remainder of discrete magnitude. Discrete magnitude is thus not just a "plurality of ones", but a plurality of which some of the ones form a limited set distinct from the rest. Since, however, the rest are subject to the same dialectic (and are also cut off by the limit that belongs to the first unity), they, too, form a discrete, limited unity – a unity that limits and excludes a further set of units that continues beyond it. This further set of units is again subject to the logic of discrete magnitude, and so on. Such magnitude proves, therefore, to be a plurality of discrete unities, each of which marks the point at which the others stop (though Hegel does not make this point explicitly until later in the text).[12]

Since the discrete unity formed by discrete magnitude has an explicit limit, and so is an exclusive one, it has the negative "edge" that was enjoyed by the qualitative one, but that was initially lost in quantitative discreteness.[13] The

many ones within this discrete unity, however, have a more ambiguous status. They initially form a unity precisely because they do not just exclude one another but are all explicitly the *same*;[14] insofar as they fall *within* that unity, therefore, they do not themselves have a negative edge. Yet the discrete unity they form *is* exclusive and is the explicit limit of other such unities; as constitutive moments *of* that unity, therefore, the many ones interrupt and limit the continuity of discrete magnitude, and so, taken together, are exclusive. The sharp difference and mutual exclusion between qualitative ones that is lost in the sphere of quantity thus begins to re-emerge in the difference between discrete, limited unities (if not within such unities themselves).

This in turn affects the way "limit" is to be understood here. This limit belongs to the discrete unity or one and keeps the latter apart from *other* such ones. Yet it is also identical with the pure *self-relation* of the one.[15] As we saw in the section on quality, the one, unlike the simple something, does not fall to one side of its limit, but completely coincides with the latter. Its pure self-relation *is* its exclusion of anything else from itself and so coincides completely with its being a limit.[16] The same is true here in the sphere of quantity. The discrete unity or one not only *has* a limit but *is* a limit itself: it is a one that excludes other such ones. In Hegel's words, "limit here is not first distinguished from the something of its determinate being, but, as the one, is immediately this negative point itself" (SL 167 / LS 212). Now the discrete unity or one, as we have seen, contains many discrete ones within itself. Since the one and its limit coincide, the latter must also coincide with the one's inclusion of the many. The limit must thus be an "enclosing, encompassing limit" that encompasses within itself the many ones of discrete magnitude.

The discrete unity formed by discrete magnitude must be an exclusive one (*Eins*), a one that shuts out other such ones. At the same time, it is a limit that encompasses many ones within itself. When quantity is conceived in this way as a discrete, limited, encompassing unity, it is what Hegel calls a *quantum* (SL 167 / LS 212). Just as being must take the form of a limited something, therefore, so quantity must take the form of a quantum – a discrete, limited unit of quantity.[17]

As we have seen, the quantum is made necessary by, and belongs to, discrete magnitude. Indeed, it is what Hegel calls "real discrete quantity" (SL 167 / LS 212). Yet discrete magnitude first acquires the limit that turns it into a quantum because it differs from and limits *continuous* magnitude. We now have to consider a consequence of this fact that has so far been overlooked.

Discrete magnitude is the active partner because *it* sets a limit to continuous magnitude which the latter otherwise lacks; in this way, as we have just explained, it distinguishes itself from its counterpart. In so doing, however, it bestows a limit *on continuous magnitude itself*. This limit, imposed on continuous by discrete magnitude, thereby comes to belong equally to both; or, as McTaggart puts it, discrete magnitude "shares" the limit with its counterpart.[18]

In this respect, therefore, the limit in discrete magnitude removes, rather than reinforces, the difference between the two kinds of magnitude: for both prove to be limited in the same way. The limit continues to reside in discontinuity and so to be the negation and limit of continuity; yet it is now revealed to be "the limit of continuity simply as such" and "the distinction between continuous and discrete magnitude is here a matter of indifference": the limit is "a limit on the continuity of *the one* as much as of *the other*" (SL 167 / LS 212). Since continuous magnitude cannot escape being limited any more than discrete magnitude can, it too must take the form of a *quantum*. A quantum of discrete magnitude thus differs not just from continuous magnitude as such but from a distinct quantum of the latter. Indeed, since a limit can be set at any point in its continuity, continuous magnitude must suffer division into multiple quanta. Accordingly, though continuous and discrete magnitude are distinct kinds of quantity, "*both* pass over into being quanta".[19]

Quantity cannot remain pure quantity, but nor can it just divide itself into two contrasting kinds; it must also, in the case of both kinds, take the form of limited units of itself, or quanta. This necessity derives, as we have seen, from the fact that in differing from continuous magnitude discrete magnitude is the direct *negation* and *limit* of the latter. The explicit difference between continuity and discreteness first gives rise to the two kinds of magnitude; the negation of continuous magnitude by discrete magnitude then leads to the emergence of a limit, and so of the quantum, in both of them.

Two things now need to be noted about the quantum before we move on. First, a quantum is a limited unity formed by many discrete ones (a "many" that is implicit in a quantum of continuous magnitude and explicit in a quantum of discrete magnitude).[20] Those ones, however, are all the *same* and for that reason constitute a continuity or unity. Thus, even though a quantum contains a plurality of ones, it is not "composed" of completely discrete atoms. This in turn means that it is not divided from the start into a fixed set of atomic units, but can be divided in many different ways. A quantum contains a plurality of units, but that plurality is not always already specified, so the quantum is in fact infinitely divisible, just like quantity as such.

Second, the ones that are interrupted to form a specific quantum can – indeed, must – continue beyond the limit set by that quantum (and, in so doing, form further quanta beyond which they continue). This is because it is the very nature of the quantitative one to be external to itself and continue beyond itself in other ones, and because such self-externality is in principle *unlimited*. A quantum thus limits a continuity that necessarily extends beyond it and that in that respect is not limited by, but is *indifferent* to, such a limit. Not only, therefore, does a quantum not contain separate atoms within itself, but it is not itself a separate "atom" of which quantity is "composed". It is a limited unit of a quantity that remains continuous both within and beyond itself.

Since quantity *must* take the form of quanta, it is, as Hegel puts it, "not indifferent to being the limit or a quantum" (SL 168 / LS 213). Yet it is inherently indifferent to, and continues beyond, any particular limit it may have. This constitutes a fundamental logical difference between quantity and quality. When quality reaches its limit it ceases being the quality it is and becomes a different one. When quantity reaches its limit, however, it continues regardless, in utter indifference to that limit. Indeed, its very nature – or "quality" – is to extend beyond, and so not to be limited by, any limit it may have.[21] This, by the way, is a subtly different point from the one Hegel makes in the remark at the start of the section on quantity. There the point is that changing the quantitative limit of something does not necessarily alter the *quality* of a thing in the way that changing its qualitative limit does: so a field can become bigger or smaller without ceasing to be a field (though, as we shall see later, there are limits to the extent to which something can survive such quantitative changes) (see SL 153 / LS 193). The point being made here is different: it is that *quantity* itself is indifferent to, and continues beyond, any limited quantum of it.

NUMBER

In quantity as such continuity and discreteness form an immediate unity, and so quantity is initially "in the *form* of one of them, of *continuity*" (SL 154-5 / LS 195). When the *difference* between the two moments is rendered explicit, quantity then gives rise to continuous magnitude and discrete magnitude. These two kinds of magnitude do not, however, simply exist alongside one another, but discrete magnitude is the negation and limit of its continuous counterpart: the point at which the latter stops. This introduces limit, and so the quantum, into both kinds of magnitude. Accordingly, in Hegel's words, "quantity is quantum, or has a limit, both as continuous and as discrete magnitude" (SL 168 / LS 213).

Yet a quantum is not just "quantity with a determinacy or limit" (SL 168 / LS 212), but is, more precisely, a limited *unit* of quantity. As a unit or one (*Eins*), a quantum is purely self-relating; as a limited unit, however, it is also negatively related to, and *excludes*, another quantum. Indeed, it excludes many other quanta from itself. A quantum, furthermore, is not an empty qualitative one, but a unit of *quantity*, and, as such, it is a continuity within itself. It is the latter because it contains (implicitly or explicitly) *many* ones that are the *same* as one another in their discreteness. A quantum is thus an exclusive one that is internally differentiated into, and encloses, a multiplicity of ones. In Hegel's words, the one as quantum is a "(α) *self-relating*, (β) *enclosing*, and (γ) *other-excluding* limit" (SL 168 / LS 213).

There is, however, an obvious tension in the idea of the quantum. On the one hand, a quantum is a limited, determinate unit of quantity and, as such, is distinct from and excludes other such units. On the other hand, all quanta are,

in the same way, *quanta* and so are not distinct after all. Insofar as a quantum is a bare quantum, therefore, it falls short of being what it is. The nature of the quantum, however, requires it to be *distinct*, for only in this way can it be the exclusive limit that it is meant to be. It must, therefore, cease being a bare quantum and take on a new form in which it is explicitly distinct and determinate.

How, then, can quanta differ from one another explicitly? They cannot do so through simply being units or ones: for, *as units*, they are all the same, just like qualitative ones. Unlike their qualitative antecedents, however, quanta are not just units, but units that encompass a manifold of other units within themselves. Since quanta as units are the same, it must therefore be the *manifold* within each one that allows it to be explicitly distinct and determinate, and so explicitly to exclude other quanta. To put it another way, that manifold or plurality (*Vielheit*) must constitute the explicit defining *limit* of each quantum – the point at which that quantum begins and another one stops. Each quantum is a unit or unity, but what makes it a truly distinct, exclusive unit is "the existence [*Dasein*] of the limit as a *plurality*" within it in its "distinction from the unity" (SL 169 / LS 213).[22]

When a quantum is thought, not just as a limited unit of quantity, but as a unit that has its explicit limit and determinacy in the plurality or manifold it contains, it is conceived as a *number* (*Zahl*). A number is thus a quantum whose limit is "*manifold within itself*" (SL 169 / LS 214). Since, however, a number's *determinacy* lies in its internal manifold, the latter cannot simply be an indeterminate aggregate. It must be a "determinate aggregate" (*bestimmte Menge*), or what Hegel calls an "*amount*" (*Anzahl*); otherwise it would not confer any determinacy on the number itself. Each number differs from every other number, therefore, by virtue of the particular amount of units it contains: 3, 4 or 5. Yet the number does not consist solely in that amount, but, as a quantum, it is at the same time a continuity or unity (*Einheit*) of ones, and so a one (*Eins*) in its own right. Accordingly, Hegel maintains, "*amount* and *unity* constitute the *moments* of number", that is, the necessary *logical* components of a number.[23] A number (*Zahl*) is not only an amount (*Anzahl*), but "the many constitute one number, *one* two [Ein *Zwei*], one ten, one hundred, and so on" (SL 169-70 / LS 214-15).[24]

In Hegel's view, a number has a distinctive logical character. It is not a pure, empty one, nor is it a bare quantum. It is a quantum distinguished from others by the specific amount of units it contains. This amount gives the number the explicit determinacy that is not yet enjoyed by the bare quantum. This, by the way, is not to say that the quantum lacks all determinacy whatsoever before – logically – it is a number. The quantum as such is already a limited, determinate unit of quantity. Its bare, "abstract" determinacy, however, leaves it looking like all the others: each quantum is *in the same way* a limited, determinate unit of quantity. The quantum becomes fully and explicitly determinate, therefore, only through the particular *amount* it contains, that is, through becoming a

number. A number is thus a "quantum in complete *determinacy*". Indeed, as Hegel later puts it, number is "the absolute determinacy of quantity" itself (SL 169, 178 / LS 213, 225).[25]

Since a number is distinct, logically, from a bare quantum and from a pure one, we might wonder whether Hegel's logic can allow there to be the *number* "one". Klaus Hartmann, for example, asks: "Can Hegel show that one is a number? Or does this arise only when one thinks of numbers as a series or calculates with them?"[26] Hegel does not address this issue directly in the *Logic*, but on the basis of what has been set out above, we can, I think, say the following. The pure one as such is not yet a number, nor would it become a number just because we calculate with it. Indeed, the pure one is not even a quantum, let alone a number, but it is an empty unit without further internal division. "One" is a number only insofar as it exhibits the logical distinction that belongs to number as such, namely that between unity and amount. The number one is thus the discrete quantitative unity whose amount consists of *one* unit, rather than two or three.

This is what "one" must be, if it is to be a number as the latter has been understood so far. In his lectures on the history of philosophy, however, Hegel makes (or is reported to make) a remark that provides support for this conception of the number "1", but that also qualifies the latter in a significant way. Hegel is discussing the system of numbers put forward by Pythagoras, but in the process he inserts thoughts of his own about number and at one point says the following:

> And indeed [for the Pythagoreans] one [*Eins*] is the principle in such a way that it is also not yet itself a number [*Zahl*], that is, not yet an amount [*Anzahl*]. Quite right, for to the number belongs: α) unity and β) amount; γ) in the one [*Eins*] both are one and the same, thus the amount is in the one only in a negative sense.
>
> —VGPW 1: 242

In this passage Hegel confirms that the one as "principle", as bare unit, is not a number, since it is not, and does not have, an amount; but he then adds that in the "one" unity and amount are "one and the same". This suggests that there is a subtle difference, for Hegel, between a pure one, in which there is no amount, and the number one, in which there is a logical distinction between amount and unity but the two completely coincide, that is, in which one unit both constitutes the amount and gives the number its unity.

One could read these lines differently and take Hegel to be simply denying that "one" is a number at all. It is noteworthy, however, that, after he has identified the logical components of number, he states, not that they are absent from the one, but that they are the same *in* the one. Furthermore, he then

claims specifically that there is an *amount* in the one, albeit in a "negative" sense. In my view, this suggests that, for Hegel, the term "one" denotes not only a bare unit, but also a *number* with an amount (and elsewhere Hegel certainly counts 1 among the "arithmetic numbers" [VGPW 1: 237]).

Yet the claim that the amount in the one is present only in a negative sense significantly qualifies the idea that "one" is a number. First, it indicates that the one, considered as a number, does not have an amount in the full sense and so is not a number in the unequivocal way 3, 10 and 100 are. The obvious reason why 1 should fall short of being a number in the full sense is that its amount is not a plurality, but comprises just one unit. Numbers proper, on this interpretation, thus begin with 2, which, as Aristotle claims, is "the *first* plurality".[27] This does not mean that 1 is not a number at all; but it must be a number in a derivative sense, for it must be derived *from* numbers proper. Second, Hegel's claim concerning the amount in the one (or 1) suggests the way in which this derivation is to be effected: the amount is present in the one in a "negative" sense because it arises through *subtracting* units from the amount in proper numbers. Logically, therefore, "one" is not originally a number; it is first of all a bare unit which is put together externally with other such units to form numbers proper (with amounts that are pluralities). By subtracting units from such a number, however, one can produce the *number* one, which differs, logically, from a bare unit by having an amount – albeit an amount in a "negative", reduced sense, since it has been reduced to just one unit.

None of this is to deny that the bare unit and the number 1 are the same *one*. They are, however, the same one conceived in subtly different ways: once as the "principle" of number and once *as* a number. When the number 1 gets added to another number, it thus becomes a bare unit in the latter like any other unit. "One" is a number with an amount, however, when it is derived, through the subtraction of units, from a number that has an amount, or determinate internal plurality, in the full sense. If this interpretation of Hegel's conception of the number 1 is correct, then the answer to Hartmann's question is clear: the number one arises, not because we calculate with numbers, but because numbers form a *series* in which they can be derived from other numbers by the subtraction of units.[28]

This conception of one as a number also suggests how, in a Hegelian account, zero can be a number: for it can be derived through a further act of subtraction and so have an amount that is "negative" because it comprises no units at all. Zero, on this interpretation, is thus no more than an empty unity or, to use different terminology, an "empty set".[29]

For Hegel, then, the inherent logic of quantity makes number necessary in the everyday sense in which we say: there are a number of things on the table. Number, so conceived, comprises a plurality of units, but its amount can be reduced to one unit, thereby yielding the number 1. "1" is thus not a number in

the full sense, like 3, 4 and 5, but, like zero, it is what we might call a derivative number: one that is produced by subtracting units from numbers proper.

Note, by the way, that numbers, for Hegel, are not just matters of convention or fictions, imposed on the world by human beings, but they are essential features of the world itself. Just as the category of quantity belongs of necessity to thought, so quantity itself belongs to the fabric of being. Quantity in turn must take the form of limited quanta, and, as we have just seen, quanta are fully determinate only as numbers. Numbers must, therefore, also belong to being: the world contains things that are intrinsically susceptible to being counted.

This is not to say that everything is equally amenable to enumeration, or that all quanta must always take the explicit form of numbers (though it is only in numbers that quanta are fully determinate). The circle, for example, can be defined as the figure in which all possible points on the circumference are at an equal distance from the centre, without specifying the length or "amount" of that distance itself (though it only becomes a particular circle when that length is specified) (SL 170 / LS 216). Furthermore, though some forms of being may in one respect be countable, they may not be best understood through numbers since quantity as such is merely a subordinate, rather than a principal, moment of them: for example, "quantity plays what we may call a more important role in inorganic nature than in organic" (EL 159-60 / 212 [§ 99 A]). In Hegel's view, therefore, the task of philosophy is not only to show the necessity of categories, such as "quantum" and "number", and their corresponding ways of being, but also to discover to which spheres of thought and being a given category most appropriately belongs – a task that is to be carried out, beyond pure logic, by the philosophies of nature and spirit (see SL 177 / LS 224).

Note, too, that even when things are completely amenable to enumeration, and indeed must be counted, number as a form of quantity remains that to which they are indifferent (at least to a certain degree). Thus, although it belongs essentially to certain things to have magnitude and to be countable, they do not necessarily come with just one number (or set of numbers) attached to them: I can, for example, increase my weight or lose my hair and still remain who I am. As we learn in the section on measure (*Maß*), there is a range of numbers within which a thing has to be confined in order to be what it is; within that range, however, the thing is indifferent to its magnitude and number. Number, for Hegel, is thus a peculiar hybrid that is both intrinsic to being *and* a matter of indifference to, and so external to, being at the same time.[30]

AMOUNT AND UNITY IN NUMBER

The logical components of a number, as we have discovered, are the amount it contains and its unity. Note that these two components coincide with *and* differ from one another. They coincide because both are constituted by the same

discrete ones. A number, as a quantum, is a unity or continuity formed by discrete ones, which form that unity insofar as they are themselves homogeneous.[31] Those ones, however, also constitute the plurality or amount contained in that unity. The same ones thus constitute both the unity of and amount in a number.

Yet there is also an immediate "qualitative difference" between the unity of the number and its amount (SL 171 / LS 217). This is because the discrete ones in a number constitute its amount only insofar as they do *not* constitute its unity (and vice versa). As just noted, the same ones are responsible for both the amount and the unity. The amount, however, differs from the unity, since it makes the number determinate whereas the unity does not (since all quanta, as unities or units, are alike) (see 2: 11). The same ones can thus constitute the one moment of number only by *not* constituting the other, and this thereby establishes a clear *qualitative* difference between the two. This difference in turn has a direct bearing on the amount in a number: for it means that the many ones, insofar as they constitute that *amount*, are not "sublated" into moments of a unity or continuity but are wholly discrete, self-relating ones.[32]

At this point, number proves to be somewhat contradictory. The ones it contains are all the same and so constitute its continuity and unity; yet, insofar as they compose the amount that defines the number, they do not form a unity but are present as wholly *discrete* ones. As such, they are the units into which the amount is not just divisible – as it would be if it were a unity – but already *divided*. This subtly alters the structure of the quantum. A bare quantum, before it is a number, contains a plurality of ones (implicitly or explicitly), but these ones are not wholly discrete because they constitute *one* unified quantum. The quantum is thus not divided from the outset into atomic units, but is, rather, infinitely *divisible* (see 2: 9). The quantum as number, by contrast, has an amount that is an aggregate of wholly discrete units. Accordingly, it "consists" of, and so is divided into, such units, even though this is exactly what quantity is not supposed to be.[33]

Indeed, this very feature of the amount is what renders it, and the number to which it belongs, fully *determinate*. A bare quantum in the abstract, as simply divisible, contains no fixed amount of units. The amount in a number, however, is divided into already existing ones. Since these ones are already given, the amount of them is fixed: in Burbidge's words, a number "includes a definite set of units".[34] Together, therefore, they constitute a wholly determinate amount and so permit the number to be this number, rather than that.

The qualitative difference between the number's amount and unity also has a bearing on that unity. Just as the amount is a determinate plurality without unity, so the unity in turn must lack such explicit plurality. The many ones constitute this unity by virtue of their homogeneity; but they do so only insofar as they do *not* form an amount. The unity they form is thus a simple unity

without explicit plurality – a unity in which the discrete units in the number are "sublated".[35]

This is not to deny that the unity of the number is a continuity and so contains the possibility of division. It is divisible, however, only into the ones that make up the *amount*, since it is those ones alone that constitute both the amount and the unity of the number: 3 is *one* number that can be divided into *three* units. The fact that the number is a continuity and unity does not mean, therefore, that it is divisible without limit: the number is divisible into the specific units that it contains and into which it is already divided.

A similar point is made by Aristotle in the *Physics*. A magnitude, such as a space or a period of time, is, for Aristotle, continuous and so is infinitely divisible (or, as he puts it, "infinite in the direction of division"). A number, however, is a "discrete" quantity and so contains a given amount of units: "number [. . .] is a plurality of 'ones' and a certain quantity of them". There is thus in number "a limit in the direction of the minimum" – that is, a limit to division – set by the amount of units that the number contains.[36] Kant, too, draws the same distinction (albeit in the context of his transcendental idealism):

> infinite division indicates only the appearance as *quantum continuum*, and is inseparable from the filling of space; for the ground of its infinite divisibility lies precisely in that. But as soon as something is assumed as a *quantum discretum*, the multiplicity of units in it is determined [*bestimmt*]; hence it is always equal to a number.
>
> —CPR B 555

For Aristotle, Kant and Hegel, therefore, numbers as such are not infinitely divisible, since each contains a given and determinate amount of units (see 1: 371).[37]

Implicit in Hegel's account of number, however, is the idea that a number is in fact infinitely divisible after all. This is not the case insofar as each is simply the number it is, for as such it is, indeed, divided into a given set of units; but numbers are infinitely divisible insofar as they can be divided *by other numbers*. In the first remark following 1.2.2.A, Hegel shows that the possibility of being divided by other numbers is an intrinsic feature of a number, and later in the main text he argues that quanta, and therefore numbers, necessarily form an infinite (in the sense of "endless") series (see SL 175, 189-92 / LS 222, 239-44). It follows that each number can be divided by indefinitely many other numbers to produce an endless series of fractions ($\frac{3}{4}$, $\frac{3}{5}$, $\frac{3}{6}$ and so on). A quantum as such (such as a length of rope) is infinitely divisible in itself; insofar as it is assigned the number 3, however, it is by definition divisible into just *three* sections; yet it thereby remains infinitely divisible, since it can be further divided by any other number we choose. (One can also conceive of a number as infinitely divisible in the following way.

Insofar as a number is a *quantum* as such, it is infinitely divisible. At whatever level one halts the division, however, one can assign a new number to the parts of the quantum that have been distinguished. This new number, as a *number*, is thus divided into the determinate amount of units it contains. As a *quantum*, however, it is once more infinitely divisible, and so on.)

The fact that the ones in the amount are wholly discrete and self-relating has a further consequence: for it means that they do not bind themselves together into an amount or a number, and so are *indifferent* to the number to which they belong.[38] That is to say, the units in 3 are not intrinsically units of the number *3*, but could just as easily belong to a different number. A number is determined by the specific amount of units it contains, but those units do not belong specifically and exclusively to that amount and that number. They are, as Hegel puts it, gathered together "externally" to form a given number, and could have been gathered differently.[39] This is an intrinsic feature of a number. It is logically necessary that there be numbers with definite amounts, but the units that comprise those amounts could equally belong to different numbers: the units in 3 and 4 could just as easily be units of 7 (or any other number).

Having said this, all the units in the amount are required for that particular amount to be what it is. An amount, Hegel insists, is not defined just by the last of its units. The number 100, for example, is determined to be that number not just by the hundredth unit it contains, but by all its units; or, to put the point another way, "each [unit] is just as much the hundredth" (SL 169 / LS 214). It is thus the whole amount in a number, not just one part of it, that constitutes the defining limit of that number.

THE INDIFFERENCE OF NUMBERS

Numbers, as we have seen, are differentiated by the amounts they contain. At the same time, Hegel now maintains, numbers are not only different from, but also utterly indifferent to, one another. The units in a number are indifferent to that number because they are wholly discrete, self-relating ones; we now see that numbers themselves are indifferent to one another. Indeed, they are indifferent *in* their very differing from one another. The reason for this is that their determinacy – their defining amount – both differs from and coincides with their unity and pure self-relation (see 2: 14–15). Let us look more closely at this feature of numbers.

As we saw in volume 1, the qualitative one, as purely self-relating, is indifferent to the others around it and to its being one of many; yet its being-*one*, or being-for-self, also actively repels and excludes other ones (and in that sense is far from indifferent to them) (see 1: 272). There is thus an essential ambiguity in the qualitative one. Similarly, there is an ambiguity in the bare quantum. The quantum, as a one, is a limit that excludes other ones; yet every

quantum, as a one, is the same as all the others and in that respect does not simply exclude them after all (2: 10–11). In the number, however, there is no such ambiguity, since there is a clear qualitative difference between its amount and its unity or being-a-one. The number sets an explicit limit to, and excludes, other numbers through its amount alone, *not* through its being a one. As a *one* or unity, therefore, it is not explicitly distinct from other numbers, but is simply a *self-relating*, *indifferent* one (just like the others). In Hegel's own words: "as one, the number remains turned back onto itself [*in sich zurückgekehrt*] and indifferent to others" (SL 170 / LS 215).

These two aspects of number, however, are not just distinct, but also coincide with one another. This, as we have seen, is because both are constituted by the same discrete ones. It is the very ones that form the amount of the number (by being wholly discrete) that also give it unity (by being the same). As Hegel puts it,

> the amount [*Anzahl*] is not a plurality *over against* [*gegen*] the enclosing, limiting one, but itself constitutes this delimitation which is a determinate quantum; the many constitute one number, *one* two, one ten, one hundred, and so on.
>
> —SL 170 / LS 215[40]

This coincidence of its amount with its being a one does not mean that a number does, after all, exclude other numbers by being a *one*. It gives the number, rather, the distinctive and contradictory characteristic that Hegel names "*being-determinate-in-itself*" (*An-sich-Bestimmtsein*). As we know from the section on quality, being determinate consists in not being what one is not, that is, in differing from, and so being related to, one's negation. A number, however, is determinate – this, *not* that – while also being a wholly self-relating, indifferent one. It is not pure, empty being-for-self, but contains a definite amount that gives it its unique determinacy; that amount, however, coincides with, and so falls completely *within*, its separate, self-relating identity. The number, therefore, must be determinate *within itself*, without reference to any other numbers. This means that 3 is defined simply by containing three units – units that it would contain even if there were no other numbers. Accordingly, as Winfield writes, "the number does not have to be contrasted with something else to be the number it is. The number limits itself through its own amount".[41] It is an independent unit with its own unique amount and so is "determinate-in-itself".

It is by virtue of this characteristic, Hegel maintains, that numbers are utterly indifferent – and external – to one another, even as they differ from one another (see SL 170 / LS 215). As ones numbers are the same, and as amounts they explicitly limit and exclude one another. As determinate within themselves, however, they differ but are thereby also purely *self*-relating. They differ from

and exclude one another, therefore, in a way that is not overtly exclusive, and consequently they simply fall *outside* one another as separate, independent ones. Their relation of difference-in-indifference thus takes the form of mutual *externality*. As Hegel puts it, a number is "a *numerical* one as the absolutely determinate, which at the same time has the form of simple immediacy and for which, therefore, the relation to another is completely external" (SL 170 / LS 215).[42]

The difference between qualitative determinacy and numerical determinacy can, therefore, be put like this. Qualitative determinacy consists in a certain immediacy, but it also requires a contrast in order to be properly determinate: a meadow has its own immediate character, but part of what it is to be a meadow is *not* to be a wood. Numerical determinacy, by contrast, consists solely in the amount contained *in* a number. The number 3 is, indeed, distinct from 4, 5 and so on; but what makes it "3" is simply the amount of units it contains *within itself*. Numbers thus have an independence and indifference to one another that qualitative things lack.[43]

In being external and indifferent to other numbers, numbers are also indifferent to the *relation* in which they stand to one another. Accordingly, they can be combined in different ways, and then separated, quite externally; that is to say, they can be *added* to, and *subtracted* from, one another (see the next chapter). Moreover any number can be combined with any other number. Qualities can, of course, also be indifferent to one another to a certain extent and so be combined without affecting either: someone can be rich and friendly or poor and friendly. Yet qualities also limit one another to a greater or lesser degree and so may not be able to be combined without being altered or destroyed: friendliness may be undermined by both richness and poverty, and if you add red to blue both are replaced by purple. Numbers, by contrast, are utterly indifferent to other numbers and can be combined without detriment to any: if you add 7 and 5, you get 12, but this is no less $7 + 5$. We saw above that the units within a given number are indifferent to the fact that they belong to that number: the units that make up 3 could just as well be put together with two more to make up 5. Now we see that numbers are also indifferent to other numbers and to being combined with them in various ways. Furthermore, Hegel insists, "this *indifference* of the number to others is an essential determination of it" (SL 170 / LS 215). Numbers are necessarily susceptible to being added and subtracted because of their distinctive *logical* structure.

Note, however, that their mutual indifference does not turn numbers into wholly discrete "atoms" of quantity. They are not such atoms, because they do not completely eliminate the continuity of quantity: quantity continues beyond any given number in the form of other numbers, just as it continues beyond any bare quanta of itself. Numbers, like bare quanta, are thus limits in the continuity of quantity.[44]

Yet numbers do not by themselves form an overarching unity – that is, a larger *number* – in the way that the units in a number form the unity *of* that number. Numbers give rise to a larger number only when they are added (or multiplied) together externally.[45] The reason why numbers differ from their internal units in this way is this. Units are, indeed, gathered together externally to form a given number, but *as* so gathered they are all the *same* in their discreteness and thereby form one continuing being, that is, one unified number. (By the same logic, bare quanta – insofar as they are not yet numbers – must also form a single unity or larger quantum.)[46] Numbers, by contrast, are all *different* in their discreteness due to the different amounts they contain; they thus remain separate from, and external to, one another and so can only be combined externally. A collection or series of identical units automatically forms a unity, but the distinct numbers 3, 4 and 5 do not form the single unity 12 unless they are added together.[47]

This, however, is not to deny that all numbers are the same in being different from one another. Yet their sameness as *numbers* does not connect them into a single unified being; rather, it gives all of them, in their difference from and indifference to one another, a common logical structure. This common structure consists in being a unity, or one, that encompasses an *amount* of units. Each number has a different amount, but each one is alike in *having* an amount (and in this respect differs from the bare quantum). The name Hegel gives to the structure exhibited by all numbers is "extensive magnitude", and it is to this that we will turn in chapter 6 of this volume.

HEGEL AND THE GREEKS ON NUMBER

Note that the numbers derived at this point in the *Logic* are the cardinal natural numbers: 2, 3 4 and so on. Fractions are not derived explicitly until we reach the idea of a direct ratio between numbers;[48] and irrational numbers cannot be conceived until we reach the idea of a quantitative infinite progress.[49] Hegel is not simply ignoring these kinds of number, therefore, but in both cases more is required to understand them than the bare idea of number as such.

Hegel's conception of a cardinal natural number clearly resembles that of the ancient Greeks – so much so, indeed, that, according to Alfredo Ferrarin, "Hegel's notion of number is peculiarly Greek in its definition".[50] Yet Hegel did not simply inherit his understanding of number from the Greeks, but he derives it in the course of his presuppositionless logic. He conceives of number in the way he does, therefore, not due to the pressure of historical tradition, but because that conception is made necessary by the very nature of quantity, and ultimately of being, itself. The proximity between the Hegelian and Greek conceptions of number shows, however, that presuppositionless logic will not

always give rise to radically new ways of understanding concepts and categories; sometimes it may confirm the insights of previous philosophers.

Carl B. Boyer contends that the Pythagoreans conceived of numbers as "collections of units" and that this conception was still preserved by Aristotle. Jacob Klein adds that a number, for the Greeks, is not just an indeterminate collection but "a *definite number*" of things or, as Aristotle puts it, a "limited plurality".[51] This conception clearly matches Hegel's idea that a number contains a definite *amount* (*Anzahl*) of units.

Furthermore, like Hegel, Greek philosophers conceived of numbers as collections of *undifferentiated* objects, or "assemblages of 'pure' units".[52] In Plato's *Philebus* Socrates acknowledges that "some arithmeticians operate with unequal units"; so, for example, "they add two armies together, or two cows" – neither of which is exactly the same – "or two things one of which might be the smallest and the other the largest thing in the world". Yet he also insists that others "would never follow their example unless every unit, no matter how many there are, is taken to be identical to every other unit", and he makes it clear that this is the philosophical way to understand numbers.[53] For the Socratic (or Platonic) philosopher, therefore (in Stewart Shapiro's words), "natural numbers are collections of pure units, which are indistinguishable from one another" – which is the case for Hegel, too.[54]

As commentators have pointed out, however, Aristotle's conception of number differs subtly from that of Plato in this respect. For Plato, numbers, as philosophers understand them, have a distinct identity of their own and are the conditions of there being numbers of ordinary objects, such as cows (and of our being able to count them); numbers are, or at least are derived from, *forms* such as "twoness" and "threeness".[55] As we have just seen, such numbers, conceived philosophically, are collections of pure units or monads; accordingly, as Klein writes, "the fact that we are able to count off a definite multitude of objects of sense is grounded in the existence of 'nonsensible' monads which can be joined together to form the number in question and which our thinking, our *dianoia*, *really* intends when it counts or calculates such things".[56] For Aristotle, on the other hand, all numbers are first and foremost the "numbers of collections of ordinary objects", rather than of pure, homogeneous units or ones.[57] Indeed, numbers are nothing apart from the cows or sheep or other things to which they relate: in Klein's words, "their being remains dependent on the being of the objects of sense".[58] Aristotle accepts, however, that when ordinary objects are counted, they are treated *as* pure, homogeneous units of enumeration. The sheep in a flock inevitably differ in certain ways, but when they are counted and assigned a number, they are treated *as if* they are all identical. It remains as true for Aristotle as for Plato, therefore, that "mathematical number consists of undifferentiated units".[59] Yet whereas Plato understands a number to encompass

units that are originally pure and identical, Aristotle understands it to collect together ordinary things that are considered *as* identical units.[60]

Since Aristotle preserves, in the way we have just described, what Klein calls the "classical definition" of number as "a multitude consisting of units", he recognizes that "the one is the source of number as number". Moreover, he sees that the one is "that by which quantity is known", since "quantity *qua* quantity is known either by a 'one' or by a number, and all number is known by a 'one'".[61] Iamblichus, Klein reports, goes further by maintaining that the one or monad is "the source of quantity" itself. Once again, therefore, the Greek conception of number (indeed, of quantity) and Hegel's conception coincide.[62]

Yet differences emerge between Hegel and the Greeks, when we turn our attention to the *unity* of the number (as opposed to the many units it contains). For Hegel, the units or "ones" in a number form a continuity or unity themselves because they are all the same: the same one (*Eins*) continues beyond itself in the other ones and so establishes a single continuing being (see SL 168 / LS 213). Insofar as the units constitute the *amount* of the number, however, they do not confer a unity on that number but are merely a determinate aggregate. There is, therefore, no distinctive unity to a number that consists in its being "two" or being "three" in particular; each number is a unity insofar as it is *one*. A number has its unity not in being one *two* or one *three*, but in being "*one* two, one ten, one hundred, and so on" (SL 170 / LS 215). Plato's conception of number is, however, subtly different. For Plato, a number becomes intelligible "as *one* assemblage of just so and so many monads" only because it is, or is derived from, a "form" (*eidos*) that is itself a unity.[63] That form, however, is not that of "unity" as such, but that of "odd" or "even", so that the units constitute an odd or even number, or that of a specific number – "two" or "three" – so that the units constitute a 2 or 3. As Klein writes, on the Platonic view "the possibility of collecting *two* monads in *one arithmos*-assemblage cannot but be understood as the effect of an original [. . .] *eidos*, be it the 'even' (*artion*), be it the eidetic 'two'".[64] Plato's conception of number thus differs in this respect from Hegel's, even though they both understand a number to be a collection of pure units.

Aristotle distinguishes his view from Plato's by rejecting the idea that a number is a unity, or one number, through a distinctive form. As Aristotle puts it in the *Metaphysics*, "as two men are not a unity apart from both, so must it be with the units" in a number: "a pair of them is nothing apart from the two".[65] There is no further *form* of "two" (or "twoness") beyond the two units themselves. In distancing himself from Plato, however, Aristotle does not thereby draw close to Hegel, for he does not accept, as Hegel does, that the discrete units in a number form a *continuity* of their own accord: number, for Aristotle, is a purely discrete, *not* also a continuous, quantity.[66] So, in Aristotle's view, how can a number, which is a collection of *many* units, be considered *one*

number? The answer, according to Klein, is this: "we comprehend a number as *one* because we do our counting over one and the same thing, because our eyes remain fixed on *one and the same thing*".[67] That "same thing" is the thing we are counting and that provides the "common measure" for the number. For Plato, we recall, a number, properly conceived, is a collection of pure units (that is identical to, or derived from, a pure form); for Aristotle, by contrast, numbers are collections of horses, cows or human beings, considered *as* units. The common measure for a number, for Aristotle, is thus "horse", "cow", "man", and so on – that is, the particular thing being counted.[68] This measure gives unity to a number because it gives the latter a single, specific focus; the number is thus not just a collection of bare units, but is the number of *these* things, rather than those, and so the number of *one* specific set of things. Whether Klein's account does justice to Aristotle's conception of number, I leave to others to judge. If it does, however, it is clear that the unity of a number, for Aristotle, is significantly different from such unity as conceived by Plato or Hegel: for an Aristotelian number lacks the explicit unity provided by a Platonic form, or generated by the discrete units in a Hegelian number, and exhibits "unity" only insofar as the things *of* which it is the number are of the same kind.[69]

CHAPTER TWO

Excursus: Hegel on the Operations of Arithmetic

As we shall see in subsequent chapters, the quantum and number will lead logically to further forms of quantity. The number will prove to be both an extensive and intensive magnitude and to be essentially subject to change; and it will also make necessary what Hegel calls "quantitative infinity" (SL 190 / LS 241). In the first remark after 1.2.2.A, however, Hegel interrupts the logical development in order to explain how the structure of number as such makes possible the fundamental operations, or "*species of calculation*" (*Rechnungsarten*), that belong to arithmetic (SL 171 / LS 216-17). As Hegel explains, his remarks about these species of calculation do not constitute a "philosophy" thereof, since they do not proceed from the further "immanent development" of the concept of number (SL 177 / LS 224). Nonetheless, they show how calculation is grounded in the features of number that have already been set out: the fact (a) that numbers, and the units within them, are external and indifferent to one another, and (b) that number contains the logical distinction between "unity" and "amount" (SL 171 / LS 217).

Hegel does not, therefore, provide what he would consider to be a philosophy of arithmetic. Yet he demonstrates that the basic calculations of arithmetic are made possible, indeed necessary, by what speculative logic shows quantity and number to be. In this sense, Hegel may be said, like Frege, to found arithmetic on *logic* – albeit a very different logic from that conceived by Frege.[1]

NUMBERING AND ADDITION

Geometry, Hegel maintains, is not as such "an art of measuring", but it *compares* spatial figures to determine whether they or their components are equal or

unequal. Yet it requires numbers (and measurement) to determine the particular magnitude of figures and lines, and it needs them to define certain figures, such as the triangle (*Dreieck*) and quadrangle (*Viereck*), at all (SL 170-1 / LS 216). By contrast, arithmetic operates specifically with numbers. The logical structure of number, Hegel claims, will thus provide the "thread" that connects the different operations of arithmetic.

The principal thing to note is that a number is an "external aggregate" (SL 171 / LS 217). This is because the units that constitute its amount are wholly discrete and self-relating, and so do not bind themselves into an amount but can be put together only externally. As Hegel puts it, a number is "an analytic figure without inner connection". In the *Encyclopaedia Logic* Hegel describes the method of speculative philosophy as "analytic", insofar as it involves "the mere *positing* [*Setzen*] of what is already contained in a concept" (EL 141 / 188 [§ 88 R]). In this case, therefore, the word "analytic" indicates that there is an intrinsic connection between the categories that arise in speculative philosophy, since each category is implicit in the one from which it is derived (see 1: 87-8). A number is an "analytic" figure, however, in precisely the opposite sense: there is no intrinsic connection between its components at all, but they are the quite abstract and mutually indifferent units into which the number can be "analysed".[2] The number is thus produced by combining such units *externally* into one amount. Similarly, numbers themselves are indifferent to one another and so can be put together only externally.

Since the components of numbers are separate units or "ones", they must be combined *one by one* in the process that Hegel calls "*counting*" (*Zählen*) or "*counting together*" (*Zusammenzählen*). Combining units into numbers, or combining the units in one number with those in another number, thus involves putting one unit and then another together with an initial unit, and so counting them off individually and thereby counting them up into one number. In Hegel's view, every arithmetical relation between numbers or their component units is external in this sense and so involves counting those units together with (or apart from) one another. These ways of relating numbers and units to one another are the different ways of *calculating* with them. All calculation, therefore, is ultimately "*counting*" and through this "the generation of numbers" (SL 171 / LS 217).[3] As we shall see, the different "species" of calculation are grounded in the fact that the numbers involved can be understood, in accordance with the two logical aspects of number, as "amounts" or "unities". Calculation can also take a negative as well as a positive form, and numbers can be generated "either by combining or by separating combinations already made" (SL 172 / LS 217).

Hegel calls the process of first generating numbers the process of "*numbering*" (*Numerieren*) (SL 172 / LS 217). This involves combining units by counting them off one by one and then breaking off at some point, thereby forming a

limited amount with just *these* units. The particular number that results is defined solely by the particular amount of units that are taken together. This amount in turn, Hegel claims, can only be "*shown*" (*gewiesen*) by referring to the units that have been counted off in thought or, perhaps, by using one's fingers. This is not to deny that each number has its own character; but that character consists solely in encompassing *this* set of units, and what constitutes this set can only be shown by counting them off individually and then stopping.

Since the numbers produced by "numbering", or counting units, are external and indifferent to one another, they can themselves be counted together to form larger numbers; that is to say, they can be *added* together. Addition is thus the first species of calculation to operate with numbers, rather than mere units (and in this respect the first form of calculation in the usual sense to be made necessary by the logical structure of number).[4] Adding numbers together, however, just involves adding their component units, or more specifically adding the units in one number, one by one, to those already combined in the other. Adding 7 and 5 thus entails beginning with the units in 7, counting in the units in 5, and then assigning a new name to the resulting amount: "12". What "12" means here, therefore, can only be shown by counting together the two sets of units and stopping; there is no other way to define it. In Hegel's view, all addition of numbers, however many there are and however large they are, involves counting together their component units. The only way to avoid "the labor of this counting" (*die Mühe dieses Numerierens*), and to prevent complicated additions becoming unmanageable, is to learn the results of basic addition "by heart" (*auswendig*) (or to use an adding machine) (SL 172, 181-2 / LS 218, 230).

KANT AND HEGEL ON "7 + 5 = 12"

In the first *Critique* Kant famously contends that "7 + 5 = 12" is a "synthetic", indeed synthetic a priori, proposition (CPR B 14-16). In his view, "the concept of the sum of 7 and 5" – expressed symbolically on the left side of the equation – "contains nothing more than the unification [*Vereinigung*] of both numbers in a single one, through which it is not at all thought what this single number is which comprehends the two of them". I can thus analyse, as much as I want, the concept of the "sum" of 7 and 5 – that is, of *unifying*, or adding together, the two numbers – but I will never find in it the number that *results* from such unification, namely "the concept of twelve". "7 + 5 = 12" is thus not an "analytic" proposition in Kant's sense. It is, rather, "synthetic", because one must go beyond the concept of the sum of 7 and 5, and actually add them in "intuition", in order to discover the number that results. In Kant's words,

> I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, so, in that image of mine [*an jenem*

> *meinem Bilde*], I now add the units, that I have previously taken together in order to constitute the number 5, one after another to the number 7, and thus see the number 12 arise.
>
> —CPR B 15-16

Or, since arithmetical equations are meant to be a priori, I see the number 12 arise by adding five points to seven, one by one, in a priori intuition.

Hegel appears to agree that "7 + 5 = 12" is not analytic in Kant's sense, when he states that the transition from the "task" of combining 7 and 5 to its "result" has nothing to do with "analysing the concept" of "7 + 5" (SL 173 / LS 219). In Hegel's view, however, this is because "7 + 5" is not a concept: "the sum of 5 and 7", as he puts it, "is the conceptless [*begrifflos*] combination of both numbers". Hegel also appears to deny, for the same reason, that "7 + 5 = 12" is synthetic in Kant's sense: in Hegel's words, "there are no concepts here, beyond which one goes". Note, however, that Hegel's critique of Kant in this regard is not immanent, for he denies that the equation is analytic or synthetic in *Kant's* sense, because it does not involve any concepts as he (*Hegel*) conceives them.

Hegel accepts that there is such a thing as the "concept of number", the distinctive logical structure that has "amount" and "unity" as its moments (EL 162 / 214 [§ 102 R]). Insofar as it is an amount, however, a number lacks any internal logical connectedness and in that sense is in fact "conceptless" (*begrifflos*): it is an "entity put together in an entirely external fashion by the arbitrary repetition of the thought 'one'" (SL 173, 181 / LS 219, 230). The concept of number is thus the concept of that which is not itself conceptual (in the way that other forms of being are, more or less explicitly). Moreover, for Hegel, there is no "concept" of specific numbers, such as 7, 5 or 12, at all, since they do not have distinctive *logical* structures of their own. As a number or amount as such, each has the conceptual structure of a "conceptless", externally constituted aggregate; as *this* number, rather than that, however, each is defined purely by its specific amount. Equally, there is no distinctive concept of the "sum of 5 and 7" beyond that of the external combination of numbers in general. Thus, not only is there no concept of "7" or "5", for Hegel, but there is no specific concept of "7 + 5" either.

In Hegel's view, therefore, the equation "7 + 5 = 12" does not take us beyond a concept and so cannot be synthetic in Kant's sense. He concedes that we can call the adding together of 7 and 5 a "synthesizing" (*Synthesieren*) (SL 173 / LS 219). Yet he notes that this "synthesizing", expressed by "7 + 5", is in fact "wholly analytic in nature", because it entails the completely external combining of two numbers and their units (and so is a merely "superficial synthesis") (SL 175 / LS 221). Hegel's use of the term "analytic" here is, of course, directly opposed to Kant's use of it, since Kant understands an analytic

judgement to be one in which one concept is "contained" in another, not one in which two elements are completely external to one another (which he calls "synthetic") (see CPR B 10-11).[5] The adding together or "synthesizing" of 7 and 5, as Hegel conceives it, is, however, related to 12 "analytically" in a sense that comes closer to Kant's: for adding the numbers, in Hegel's view, just *is* the process of counting off the units one by one and producing the total 12. This is not to say that we find the "concept" of 12 by analysing the "concept" of "7 + 5", since, strictly speaking, for Hegel, there are no such concepts. Yet nor do we have to go *beyond* "7 + 5" to discover that it is equal to 12: for the combining of 7 and 5 is *itself* the process of "counting from seven onwards until the five ones have been exhausted" and thereby forming the number 12 (SL 173 / LS 219). In this respect, "7 + 5 = 12" is, for Hegel, an *analytic* expression in a roughly Kantian sense.

It is easy to get lost here amid the different senses of "synthetic" and "analytic", so to recapitulate: in Hegel's view, "7 + 5 = 12" is, strictly speaking, neither a synthetic nor an analytic expression in Kant's sense, since it involves no concepts (as Hegel understands them). Adding 7 and 5 can, however, be regarded as "synthesising" them – though such synthesising is itself "analytic" in one Hegelian sense, since it entails adding the numbers together externally. "7 + 5 = 12" is also an analytic expression in a roughly Kantian sense, since adding 7 and 5 just *is* the producing of 12.[6]

Yet one might point out on *Kant's* behalf that he actually has two different conceptions of "7 + 5" and I have presented Hegel's response to just one of these. Kant maintains that "the concept of the sum of 7 and 5 contains nothing more than the unification of both numbers in a single one", and that the thought of such "unification" does not itself contain the thought of the number that results (CPR B 15). Hegel responds to this by insisting that there is no clear distinction between the *unifying* (expressed on the left side of the equation) and the *result* (expressed on the right), because the former just is the generating of the latter. Yet Kant also conceives of "7 + 5" in a slightly different way, for he writes at one point that what is thought in the "concept of a sum = 7 + 5" is the idea "that 7 *should* be added to 5", rather than the idea of actually adding the two (CPR B 16). On this conception of "7 + 5", it seems more obvious that one has to go beyond the mere "should" to reach the result of the addition (and that "7 + 5 = 12" is thus a synthetic proposition).

Hegel appears to recognize that "7 + 5" is understood by Kant in this way, when he writes, later in the *Logic*, that it contains the "*demand*" (*Forderung*) that 7 and 5 be combined (SL 704 / LB 246). He immediately points out, however, that "7 + 5" contains the demand that they be combined "into *one* expression"; in other words, the thought that 7 and 5 *should* be combined is the thought they should actually *be* combined *to form one amount*. Since the act of combining 7 and 5 simply involves counting off seven ones and then continuing

to count off five more, the resulting amount is automatically 12. Accordingly, the demand that 7 and 5 be combined "into *one* expression" just *is* the demand that they be added together *to make 12*. For Hegel, therefore, even though there may appear to be a notional difference between the thought that 7 and 5 should be combined (expressed by "7 + 5" or "5 + 7") and the number that is their combination, namely 12, it is in fact the case that "5 + 7 and 12 are absolutely the very same content". The equation "7 + 5 = 12" is thus, from a Hegelian perspective, an *analytic* expression in a sense that is similar, though not identical, to that given to the term "analytic judgement" by Kant.

MULTIPLICATION AND "RAISING TO A POWER"

Insofar as numbers are simply to be added together (or subtracted from one another), they are treated as immediate numbers that are indifferent to whether they are equal or unequal. That is, they are treated as "contingent" magnitudes that may be, but are not necessarily, the same. As not necessarily equal (though not necessarily unequal either), they are just various given numbers and so can be said to be "*unequal* in general" (Ungleiche *überhaupt*) (SL 172 / LS 218).

Yet numbers can also be added insofar as they are specifically *equal*: we can add 2 and 2 and 2. In this respect, Hegel notes, each number is no longer just a given number, but can be regarded as the same "unity" or "unit" (*Einheit*) that is taken a certain *amount* of times. Addition conceived in this way is, of course, multiplication. Multiplication is thus an operation with numbers – in the case above, 2 and 3 – that correspond to the two logical moments of number as such, "unity and *amount*": the unit 2 is taken *3* times (SL 175 / LS 222). Multiplication is still a form of addition that involves counting together the units *in* numbers, and so it is still grounded in the fact that such units, and the numbers to which they belong, are external to one another. It is, however, also grounded in the qualitative difference between unity and amount that characterizes number as such. Multiplication is thus derived from number conceived in a fuller sense than in the case of simple addition.

Hegel points out that in multiplication either number can be taken as the unity and either as the amount: taking the unit 2 three times is the same as taking the unit 3 twice. In division, too – which is the "negative species of calculation" associated with multiplication – either number can be taken as the unit or amount.

> The divisor is determined as unit and the quotient as amount whenever the stated task of the division is to see *how many times* (the amount) *one* number (the unit) is contained in a given number; conversely, the divisor is taken as amount and the quotient as unit whenever the stated intent is to divide a

number into a given amount of equal parts and to find the magnitude of the part (of the unit).

—SL 175 / LS 222

Now in the number as such the moments of unity and amount not only differ logically, but also coincide, since they are constituted by the same units (see 2: 14-15). Such coincidence finds explicit expression in multiplication when the numbers taken respectively as unit and amount are themselves equal. In simple multiplication, one number (the unit) is taken a certain amount of times, but the amount of the unit and the amount of times it is taken are not necessarily the same: so 2 can be taken 3 or 4 or 5 times and so on. When, however, the amount in the unit is *equal* to the amount by which it is multiplied, "the progression to the equality of the determinations inherent in the determination of number is completed" (SL 175-6 / LS 222). In this case, 3 is multiplied by 3, 4 by 4, and so on; that is to say, the number is squared. The third species of calculation grounded in the logical character of number is thus squaring or "*raising to a power*" (*Potenzieren*) (with its negative counterpart, the extraction of a root) (SL 176 / LS 223).

Later in the *Logic* Hegel will argue that quantity – and more specifically the "infinity" of the quantum – makes necessary the *ratio* that holds between a number and its second power (as well as the direct and inverse ratios) (see SL 278-9 / LS 359-61). In the remark we are considering, however, Hegel derives raising-to-a-power as a further form of multiplication, and thus of addition, and thus of counting, from the logical structure of the number as such. Raising-to-a-power is simply the process of counting, in which the numbers counted together are the same and so constitute the unit, and the amount of times the unit is taken is the same as the amount in the unit itself (see SL 176 / LS 223).

Hegel claims that there are no further determinations in the number as such that would make further species of calculation necessary. All there is, is (1) the mutual externality and indifference of numbers and their component units, which makes counting them together, or *addition*, necessary (as well as subtraction); (2) the logical difference between unity and amount, which makes *multiplication* necessary (as well as division); and (3) the coincidence or identity of unity and amount, which makes *raising-to-a-power* necessary (as well as extracting a root). One can, of course, raise numbers to higher powers than the second, but this operation is understood by Hegel to be merely a "*formal* continuation" or "repetition" of the original operation of raising-to-a-power and so does not count as a different species of calculation (SL 176 / LS 223). Differentiation, on the other hand, can be regarded as a different kind of calculation, since it involves a distinctive technique in addition to multiplication and raising-to-a-power. In Hegel's view, however, it requires for its explanation and justification the idea of quantitative infinity as well as that of a "qualitative

ratio" of "quantitative determinations", and so is not based on number as such.[7] Number as such, for Hegel, grounds only the operations we have considered.[8]

The fact that these operations have their ground in number has, however, one further consequence. Since numbers are mere aggregates of units, put together externally, no special judgement or speculative comprehension is required to add or multiply them together. One simply has to count off one by one the units in the numbers to be added and remember the result, or to draw on results one has already learned by heart if one is to perform more complex calculations. Arithmetical calculation is in this sense "thoughtless" and "mechanical", and requires no special powers of insight (SL 181 / LS 230). This is not to say that it does not demand significant "effort" (*Kraftanstrengung*) on the part of the one doing the calculating; the effort consists, however, in holding fast what is merely external and "without concept" (*Begriffloses*) and combining it in an external, non-conceptual way. This means in turn that *machines* can do the work of calculation, as well as human beings can. Mechanized calculation, performed by a calculator or computer, is thus not just a clever invention by humans to make life easier; it is made possible by the very nature of number, and thus of quantity, itself.

CHAPTER THREE

Excursus: Hegel and Frege – Similarities and Differences

Hegel's conception of number resembles, though does not simply replicate, the Greek conception(s). In the late nineteenth century, however, Gottlob Frege developed what many today consider to be a more profound and fruitful conception of number than is found among the Greeks (and, by implication, in Hegel's *Logic*). Before we proceed with the logic of quantity, therefore, we need to consider whether Hegel's conception of number is, indeed, superseded by Frege's conception, or whether – as I will argue – Frege's thought itself rests on assumptions that Hegel calls into question.

According to Charles Parsons, Frege's work in the philosophy of mathematics "represents an enormous advance in clarity and rigor" over that of his predecessors. "Frege's analysis increases our understanding of the elementary ideas of arithmetic" – such as number – and "there are fundamental philosophical points that his predecessors grasped very dimly, if at all, which Frege is clear about".[1] Gregory Currie goes further and maintains that Frege's *The Foundations of Arithmetic* (*Die Grundlagen der Arithmetik*) (1884) is "perhaps the most important work ever in the philosophy of mathematics".[2]

Yet Frege, for those influenced by him, is much more than a philosopher of mathematics. Michael Dummett famously credits him with instigating the "linguistic turn" that led to the emergence of analytic philosophy as a whole, and he claims that the *Foundations* "may justly be called the first work of analytical philosophy".[3] Some, like Adrian Moore, stop short of making such grandiose claims; but Moore agrees that "Frege is of colossal

significance to the analytic tradition", even if he was not necessarily its principal instigator.[4]

It is clear, therefore, that for many, especially in the analytic tradition, the merits of Hegel's conception of number will depend on how well it compares with that put forward by Frege. It is equally clear, however, that from Frege's perspective Hegel's conception cannot compare well at all, since it closely matches the very one that Frege explicitly rejects, namely, the conception of number as "a set of featureless units". Moreover, in Dummett's view, Frege does not just reject this conception, but he does so "brilliantly, decisively and definitively".[5] There would appear, therefore, to be nothing in Hegel's account of number to interest Frege and those who follow his lead. Indeed, as Moore puts it, there is "a great deal in Hegel" – over and above his conception of number – "that, from Frege's perspective, will seem not to make any sense at all".[6] So is there really any point in comparing Hegel and Frege on number? Don't we already know that, for most of those, such as Bertrand Russell, who see Frege's work as the foundation of future philosophy, there is little to be gained from studying Hegel's views on anything? After all, didn't Russell proudly proclaim that "almost all Hegel's doctrines are false";[7] and isn't Moore right to note that Frege himself would have considered Hegel's conception of dialectical reason, with its openness to contradiction, to be evidence of "a hitherto unknown type of madness"?[8]

There are contemporary philosophers, such as John McDowell and Robert Brandom, who have drawn inspiration from both Frege and Hegel, so the latter do not have to be absolutely opposed to one another.[9] Indeed, I will suggest myself below that there are some similarities between them. Yet, as far as I know, neither McDowell nor Brandom has indicated that there is anything significant to be learned from Hegel's conceptions of quantity and number in particular. Neither provides evidence, therefore, that post-Fregean philosophers more generally can find merit in Hegel's account of number.

My interest, however, does not lie principally in examining and evaluating Hegel's account of number in light of the standards set by Frege. It lies, rather, in highlighting what Hegel would regard as the dogmatic assumptions underlying Frege's thought. As we shall see, these assumptions underlie Frege's critique of the traditional, Greek conception of number, his own conception of number and, indeed, his conception of logic as such. By contrast, as I have argued throughout this study, Hegel's speculative logic is radically presuppositionless in resting on no determinate assumptions about thought or being. The merit of Hegel's conception of number is in turn that it is derived in the course of such presuppositionless logic. Hegel's logic – together with the distinctive conception of quantity and number to which it gives rise – is thus evidence, not of "madness" on Hegel's part, but rather of a mode of thinking that is considerably more self-critical and less question-begging than Frege's.

Many will no doubt dismiss the idea that Hegel's logic is presuppositionless, and many have dismissed it in the past (from both the "analytic" and "continental" traditions), because they assume – more or less dogmatically – that all thought rests on assumptions (whether these be Fregean "axioms" or Gadamerian "prejudices"). Yet if it can be shown that Frege's "revolutionary" conception of number (and, indeed, of logic as such) in fact takes much more for granted than Hegel's conception of number (and of logic), then perhaps philosophers drawn to post-Fregean ways of thinking can be persuaded to take Hegel's conception of number, and of quantity altogether, more seriously than they have done hitherto.

This study has endeavoured to show that, and how, Hegel derives his conception of number and quantity, via quality, from the indeterminate being with which presuppositionless logic must begin; and readers can judge for themselves whether that derivation is successful. The task now is to show that Frege's conception of number, in contrast to Hegel's, is very far from presuppositionless.

I will focus primarily on *The Foundations of Arithmetic* (GA), which is hailed by Dummett as "Frege's masterpiece".[10] In the years following the publication of the *Foundations*, however, Frege made significant changes to his philosophical logic: he subtly modified his conception of the "extension" (*Umfang*) of a concept, and he introduced the idea of a "value-range" (*Wertverlauf*) as well as the now famous distinction between "Sinn" and "Bedeutung". These changes then informed his *Basic Laws of Arithmetic* (*Grundgesetze der Arithmetik*) (1893, 1903).[11] Furthermore, in that text Frege presented formal proofs of certain fundamental theorems concerning numbers and so demonstrated (to his satisfaction) that arithmetic is a priori and "analytic" in his distinctive sense, whereas in the *Foundations*, by his own admission, he merely "made it probable [*wahrscheinlich*] that arithmetical laws are analytic judgements and therefore *a priori*" (GA 119 [§ 87]).[12] For these reasons, Dummett remarks, one might think that the *Foundations* "should be set aside as a brilliant but immature work" and that we should study Frege's philosophy of arithmetic primarily from his *Basic Laws*. We cannot do that, however, Dummett argues, because it is in the *Foundations*, rather than the *Basic Laws*, that Frege provides the philosophical justification for his theory of natural numbers (as opposed to formal proofs of theorems concerning them).[13] Moreover, despite the changes in Frege's philosophical logic just mentioned, the "logical construction of the theory of the natural numbers" set out in the *Foundations* remains essentially unaltered in the *Basic Laws*. Dummett concludes, therefore, that "we may consider *Grundlagen* [*Foundations*] as expressing, with fair accuracy, Frege's mature philosophy of arithmetic, not merely a superseded phase of his thinking".[14] I shall follow Dummett's lead in this respect and focus primarily (though not exclusively) on the *Foundations*.

By the way, in my discussion of Frege I shall touch only briefly on "Russell's paradox", which might be thought to undermine Frege's conception of number.[15] Frege was initially convinced that he could remove the paradox, but it eventually led him to abandon his twenty-five year project to prove that the truths of arithmetic are analytic.[16] Russell, however, appears to have regarded it as less damaging to Frege's project than Frege came to believe; and, more importantly for our purposes, he did not think that it undermined Frege's conception of number. In his *Introduction to Mathematical Philosophy* (1919), for example, Russell writes:

> The question "What is a number?" is one which has been often asked, but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his *Grundlagen der Arithmetik*.[17]

I shall leave it to others to judge whether Russell is right to suggest that Frege's conception of number survives the paradox. Russell, however, is by no means alone in believing that it does survive and that it continues to be of considerable importance and influence (even if it requires some modification), and that suffices to justify my examination of that conception here.

FREGE'S PROJECT

According to Joan Weiner, Frege's overall aim is "to give gapless logical proofs of the basic truths of arithmetic". More specifically, it is "to define the individual numbers and the concept of number and to show that all the truths of arithmetic can be proved using only these definitions and the general laws of logic".[18] In this way Frege aims to show that the truths of arithmetic are *a priori* and *analytic* and thereby to provide the most general justification for them.

Frege takes over the distinction between analytic and synthetic truths from Kant, though he conceives of the terms "analytic" and "synthetic" in a slightly different way from Kant himself; or, rather, he builds on one aspect of Kant's conception of them, while setting another aspect to one side. In the first *Critique*, Kant says that "in all judgments in which the relation of a subject to the predicate is thought [. . .] this relation is possible in two different ways" (CPR B 10). Either the predicate is "contained" in the subject or it lies "outside" the latter. "In the first case", Kant writes, "I call the judgement **analytic**, in the second **synthetic**". Thus the judgement "all bodies are extended" is analytic, since the concept of extension is already contained in the concept of a body, whereas the judgement "all bodies are heavy" is synthetic, because (for Kant) heaviness is not built into the concept of a body (CPR B 11).[19]

On the basis of this distinction Kant claims that a "mathematical judgement" or equation, such as $7 + 5 = 12$, is synthetic. For Kant, this equation is a priori,

rather than a posteriori, since it is necessary; but it is synthetic because “the concept of twelve” is not already contained in “the concept of the sum of 7 and 5” (CPR B 14-15). In Kant’s view, the latter concept (which we can think of as the subject of the judgement) “contains nothing more than the *unification* [*Vereinigung*] of both numbers in a single one”; but in that concept “it is not at all thought what this single number is which comprehends the two of them”. In other words, the subject, or left side of the equation, is the thought of the *operation* of adding 7 and 5, but that thought by itself does not tell us what specific number *results* from the operation: we have actually to carry it out (in a priori intuition) in order to “see the number 12 arise” (CPR B 16; see also B 205). Frege and Hegel, and indeed many others, reject Kant’s claim that mathematical equations are synthetic, but one can see why he conceives of them as he does.[20]

Note that, for Kant, subject-predicate judgements, including equations, connect *concepts* with one another (where a “concept” is understood as a form of discursive thought). In non-mathematical cases, the subject of such a judgement may well refer to what is given through sensuous intuition, but a concept is nonetheless needed in order to cognize the latter as an object. The judgement thus connects the predicate-concept directly with the subject-*concept* and only indirectly with what is given in intuition. So in the analytic judgement “all bodies are divisible” the concept of the divisible is related to the *concept* of body, which is in turn related to “certain appearances that come before us” (CPR B 93).

As we have just seen, in a synthetic judgement the predicate-concept is not contained in the subject-concept itself. What then justifies the judgement? In Kant’s view, it must be either empirical experience, or in the case of mathematics a priori intuition, or in the case of the principles of pure understanding the fact that they set out the necessary conditions of objects of experience (see CPR B 11-16, 193-7, 264). In an analytic judgement, by contrast, the predicate *is* contained in the subject (implicitly or explicitly); no experience or intuition is needed, therefore, to justify the judgement. When the subject is something empirical, such as gold, experience will, of course, be needed to formulate the concept of it in the first place; but once that concept is in place no further experience is required to ground an analytic judgement about it (such as “gold is a yellow metal”). As Kant puts it, “I already have all the conditions for my judgment in the concept [of the subject], from which I merely draw out the predicate in accordance with the principle of contradiction” (CPR B 12). This remark is important because it reveals that, for Kant, an analytic judgement actually has a twofold justification: it is justified by the concept of the subject *and* by the fact that it would be contradictory to deny that the predicate belongs to that subject. Whereas an empirical synthetic judgement is justified by experience, therefore, an analytic judgement is justified (at least in part) by a *principle of logic*: the principle of (non-)contradiction.[21]

In a text entitled "Comments on *Sinn* und *Bedeutung*" (1892-5) – a sequel to the famous essay on the topic – Frege sets himself apart from Kant by arguing that we should dispense with the logical distinction between *subject* and *predicate* which Kant takes for granted. "It would be best to banish the words 'subject' and 'predicate' from logic entirely", Frege writes, "since they lead us again and again to confound two quite different relations: that of an object's falling under a concept and that of one concept being subordinated to another" (FR 175 / ASB 28). We will consider Frege's distinction between "object" and "concept" in more detail in the next chapter. At this point we need note only that Frege distinguishes between two kinds of judgement – those that relate an *object* to a concept and those that relate a *concept* to a concept – and that, in his view, this distinction is obscured when we simply analyse judgements into their subjects and predicates. For Frege, the judgement "Cato is mortal" subsumes the object "Cato" under the concept "being mortal", whereas the judgement "all humans are mortal" subordinates the concept "being human" to that of "being mortal" (see LM 108). Both judgements do, indeed, have a grammatical "subject" and "predicate"; but, logically, they do not both connect a subject- and a predicate-*concept*, since, logically, the first one does not express a relation between two concepts at all. As Frege remarks in the *Foundations*, "what if the subject is an individual object? [. . .] Here there can be no talk at all of a subject-concept [*Subjektsbegriff*] in Kant's sense" (GA 120 [§ 88]). This means, however, that Frege cannot take over Kant's distinction between analytic and synthetic judgements in an unmodified form: for he cannot accept the view that in all such judgements a predicate-concept either is or is not "contained" in a subject-concept.

Strictly speaking, indeed, on Frege's understanding, there is not even a "subject-concept" in judgements that subordinate one concept to another. This is because (in the *Begriffsschrift* [1879]) Frege recasts such judgements using what has come to be called the "universal quantifier": "for any x" (albeit in his distinctive notation).[22] In the *Foundations* Frege points out that the judgement "all whales are mammals" initially appears to be about animals, rather than concepts: it seems to be about all the whales there are (GA 82 [§ 47]).[23] In fact, however, so he contends, it is about concepts and subordinates the concept "being a whale" to that of "being a mammal". Yet, if we follow the *Begriffsschrift*, the judgement does not contain a *subject*-concept. This is because, for Frege, the proper form of the judgement is: "for any *x*, if *x* is a whale, then *x* is a mammal". (The judgement "*a* whale is a mammal" is to be recast in the same way.) In this rendering of the judgement the two concepts concerned are, indeed, related to one another, but neither, clearly, is the subject of the judgement. The subject, if one can speak of such a thing, is "*x*" or "whatever may be substituted for *x*"; that is, the subject is the variable that stands for any *object* we like (see FR 70 / BS 20 [§ 11]).[24] The judgement does not, therefore, connect a subject-concept with a predicate-concept in the way Kant describes.

As we will see, this is true, for Frege, even when we formulate certain judgements explicitly about *concepts* as opposed to objects – when we say, for example, that "the concept 'horse' is a concept easily attained" (FR 184 / ÜBG 50). In this case, we are clearly talking about a concept and not an object: we are saying, not that a *horse* is easy to get hold of, but that the *concept* "horse" is easy to acquire and understand (in contrast, perhaps, to the concept "number"). Yet Frege insists that when we judge in this way we actually turn "the concept 'horse'" into an *object* – the object about which the judgement is being made (FR 184-5 / ÜBG 50-2). So such judgements, for Frege, have objects, rather than concepts as such, as their "subjects" and subsume such objects under concepts; and this in turn means that they can be neither synthetic nor analytic in precisely Kant's sense.

Equally important for our purposes is Frege's denial that judgements of identity, including mathematical equations, hold between *concepts*. As he writes, "the relation of equality, by which I understand complete coincidence, identity, can only be thought of as holding for objects, not concepts" (FR 175 / ASB 28). This is not to say that *different* objects can be equal to one another – for, as we shall see, every object for Frege is unique – but that different object-expressions can denote the *same object*. The equation "7 + 5 = 12" thus cannot be synthetic *or* analytic in Kant's sense, because it does not relate two *concepts* to one another, but states that the *object* denoted by the sign "7 + 5" is the same as the *object* denoted by the sign "12" – the object being the number 12 itself. Frege makes this point clear in the essay, "Function and Concept" (1891):

> The expressions "2", "1 + 1", "3 – 1", "6 : 3" all have the same *Bedeutung*, for it is quite inconceivable where the difference between them could lie. Perhaps you say: "1 + 1" is a sum, but "6 : 3" is a quotient. But what is "6 : 3"? The number that when multiplied by 3 gives the result 6. We say "*the* number", not "a number"; by using the definite article, we indicate that there is only a single number. Now we have: (1 + 1) + (1 + 1) + (1 + 1) = 6, and thus (1 + 1) is the very number that was designated as (6 : 3). The different expressions correspond to different conceptions and aspects, but nevertheless always to the same thing [*derselben Sache*].
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> —FR 132 / FB 4

It is in this essay, by the way, that Frege first introduces the technical distinction between "Bedeutung" and "Sinn" that is further developed in his more famous essay on the topic (see FR 138 / FB 10). The *Bedeutung* of an expression is that to which the expression refers – as opposed to the *way* in which the *Bedeutung* is presented by the expression, which is the expression's "sense" (*Sinn*) – and it can be an object or thing (in the case of an object-word)

or a concept (in the case of a concept-word).[25] In the passage just cited, however, the *Bedeutung* is clearly an object, *not* a concept: it is the number denoted by the expressions "2" and "1 + 1" and so on, and, for Frege, every number is an *object* (see GA 71, 90 [§§ 38, 57]).

Although the *Foundations* was written several years before "Function and Concept", the idea that equality holds of objects, rather than concepts, is found there, too. In § 62, for example, Frege states that number words stand for "independent objects" (*selbständige Gegenstände*) – the numbers – and that there must therefore be "a class of propositions" in which we "recognize" a number as the "same again" (*wiederkennen*). Such propositions, Frege continues (in § 63), are those that implicitly or explicitly express an "equation" (*Gleichung*) between numbers (GA 94-5). Judgements of equality, such as mathematical equations, thus hold of objects, rather than concepts (and state that the objects denoted by two different expressions are in fact the same object).

Now, as noted above, Kant understands mathematical equations, such as "7 + 5 = 12", to be synthetic. Frege, by contrast, thinks that all "arithmetical truths", including equations, or mathematical "formulae", and the principles or "general laws" of arithmetic, are *analytic* (GA 29, 119, 138 [§§ 4, 87, 109]). *Pace* Adrian Moore, however, Frege did not, and could not, intend "nothing other by analyticity than what Kant intended",[26] because he could not understand mathematical equations – or, indeed, any analytic judgements – to be those in which a predicate-*concept* is "contained" in a subject-*concept*. Frege's conception of analyticity is, however, indebted to Kant's in another respect: for Frege takes over, and extends, Kant's idea that analytic judgements are justified by a *principle (or law) of logic* (in Kant's view, the principle of contradiction).

Frege insists that "one must distinguish mathematical formulae, such as 2 + 3 = 5, that deal with specific numbers from the general laws that hold for all whole numbers" (GA 29 [§ 5]). An example of the latter is the law of associativity that states: a + (b + c) = (a + b) + c (GA 25, 31 [§§ 2, 6]). For Frege, however, all such formulae and laws are analytic, since they can be traced back to, and so are ultimately justified by, "general logical laws and definitions" (GA 27, 36 [§§ 3, 9]).[27] By contrast, synthetic judgements require "truths that are not of a general logical nature" for their justification, namely truths based on experience or intuition (GA 27-8 [§ 3]).[28]

Frege sets out what he takes to be the most fundamental general logical laws, or "axioms" of thought, in part 2 of the *Begriffsschrift*. These laws rest on the assumption that *modus ponens* is the basic rule of inference, and they are constructed from the conditional ("if a, then b") and negation as primitive connectives, as well as the universal quantifier ("for any a") and identity, though not all of these are used in each law.[29] The laws include, for example, "if a, then

'if b, then a'" (Frege's axiom 1); "if 'if b, then a', then 'if not a, then not b'" (axiom 28); the law of contradiction "if a, then not not a" (axiom 41), and the law of identity "a = a" (axiom 54).[30]

My concern here, however, is not specifically with Frege's account of these laws; nor will I examine how he uses these laws to derive the general laws of arithmetic. My focus will be on the other source of the laws and equations of arithmetic, namely the definitions of *number* (*Anzahl*) as such and of the specific numbers (*Zahlen*) (GA 25 [§ 2]).[31] In Frege's view, the task of proving the laws of arithmetic and mathematical equations to be analytic requires deriving them from the laws of logic *and* the definitions of number and the numbers. More precisely, the general laws of arithmetic are to be derived from the laws of logic and the definition of number as such, and equations are then to be derived from these general arithmetical laws and the definitions of the individual numbers (see GA 36, 47-8 [§§ 9, 18]). The specific project of defining number and the numbers is thus central to Frege's overall project of proving arithmetic to be analytic. This indeed, he tells us in the *Foundations*, "is the task of the present work" (GA 28 [§ 4]).[32]

Frege's comments on "definition" in the *Foundations* can, however, appear confusing. He talks at one point of "the definability [*Definierbarkeit*] of the concept of number" (GA 29 [§ 4]), but he also appears to suggest that concepts as such cannot be defined. Certain concepts, he says, are in need "of a sharper determination" (*einer schärferen Bestimmung*) (GA 25 [§ 1]), but a definition, he insists "fixes the meaning of a *sign* [*Zeichen*]" (rather than a concept) (GA 99 [§ 67], emphasis added).[33]

The same claim is made in Frege's unpublished article "Logic in Mathematics" (1914): "a sign has a *Bedeutung* once one has been bestowed on it by definition" (FR 314 / LM 100). This text was written thirty years after the *Foundations*, but Weiner contends that "Frege's views on definition did not change" from the earlier to the later text.[34] If we follow Weiner, we can thus draw on "Logic in Mathematics" to clear up the appearance of confusion generated by the *Foundations*.

In this later text Frege makes it clear that a definition is a "stipulation" (*Festsetzung*) and that it is "really only concerned with signs" (FR 313 / LM 99). More specifically, a definition stipulates that a simple sign shall have the same sense and *Bedeutung* as a certain group of other signs or a complex expression, and in this way the definition fixes the *Bedeutung* of that simple sign. The task of the *Foundations*, therefore, will be to fix the *Bedeutung* of the sign or word "number" (*Anzahl*) by identifying a set of other words whose *Bedeutung* the word will share (and then to show how one can do the same with specific number-signs, such as "1", "2" and so on). Frege maintains in the *Foundations* that "for us the concept of number has not yet been fixed", and that it will be fixed "by means of our explanation [*Erklärung*]", that is, by

defining the *word* "number" (GA 95 [§ 63]). It is in this sense, therefore, that the concept of number will itself be "defined".

Frege recognizes, however, that the word "number" is not completely new and unfamiliar to us. He will thus draw on the familiar meaning of the word in order to find a complex expression through which the word will be given its precise definition (see 2: 114–15, 120–1). Once that definition has been given, it can then be understood as no longer just a stipulation but a statement of *identity* between the *Bedeutung* of the word "number" itself and that of the complex expression through which the latter was defined. That is to say, the definition can be given the form "A = B", and, as Michael Resnik points out, "it can then be used as a premise in proofs as if it were an axiom".[35] In this way, the logical definitions of both number as such and of specific numbers can form the basis, together with the general laws of logic, from which we *prove* the general laws of arithmetic and mathematical formulae, such as 7 + 5 = 12, and thereby demonstrate that such laws and formulae are *analytic* in Frege's sense.

As noted above, however, my purpose here is not to examine how Frege proceeds with such proof. It is to examine the way in which he defines number as such and the specific numbers, and to consider his definitions in light of Hegel's account of number in the *Logic*. Before I do this, though, I will first highlight in the rest of this chapter various more general similarities and differences between Hegel and Frege. In the next chapter I will then point to what, from a Hegelian perspective, are the unjustified assumptions underlying Frege's thought; and in the following chapter I will focus directly on Frege's account of number (and the numbers) in the *Foundations*.

SIMILARITIES AND DIFFERENCES BETWEEN FREGE AND HEGEL

Adrian Moore is surely correct when he writes that Frege and Hegel "can easily seem worlds apart".[36] Indeed, in certain respects they *are* worlds apart, as we will see. Yet there are some similarities between them that are worth noting. (In noting these, by the way, I will draw on works written by Frege before and after the *Foundations*, when these help to illuminate positions he takes in that text itself.)

Logicism

First, like Frege and unlike Kant, Hegel thinks that mathematical equations, such as 7 + 5 = 12, have a purely *logical* ground. As we saw in the last chapter, that ground, for Hegel, is the logical structure of number derived in the *Logic*, rather than the definitions and logical "laws" that Frege takes to be fundamental. Furthermore, Hegel does not start out, as Frege does, with the specific aim of deriving arithmetic from logic;[37] he has no aim beyond discovering what (if anything) is inherent in, and entailed by, pure being. In showing being to entail

quantity and number, however, Hegel also shows that the core operations of arithmetic, such as addition, arise directly from "the conceptual determination [*Begriffsbestimmung*] of number itself" (SL 171 / LS 216-17). Hegel and Frege are thus different thinkers with different projects (and different conceptions of logic). Nonetheless, despite these differences, they share the view that logic grounds arithmetic, and so in that sense, and in contrast to Kant, they can both be considered "logicists".

The objectivity of logic

The second similarity between Hegel and Frege is that both understand logic and logical entities, such as numbers, to be *objective* and to have a necessary structure of their own. In the *Foundations*, Frege states that what is objective is "what is independent of our representations [*Vorstellungen*] and the like" (GA 57 [§ 26]). Yet he also insists that it is to be distinguished from "the handleable [*das Handgreifliche*], spatial, actual". The objective, he continues, is not what actually exists in space and time, but it is "the law-governed [*das Gesetzmäßige*], conceptual, judgeable": it is the logical and rational itself (GA 58 [§ 26]). Moore contends that "the subject matter of logic" – the logical – is, accordingly, "transcendent for Frege".[38] This claim needs to be treated with caution, however, for Frege maintains that what is objective is not completely *separate* from us. "By objectivity", he writes, "I understand an independence from our sensing, intuiting and representing [. . .] but not an independence from reason [*Vernunft*]" (GA 59 [§ 26]). The logical and conceptual is thus not just "out there" in complete indifference to us, like the stars and planets, but it is the objective pattern of necessity that governs our reason and that the latter thus comprehends as "its ownmost" (*ihr Eigenstes*) (GA 135 [§ 105]).

While objective, therefore, logic, like arithmetic, directly governs *our* judgement and inference, rather than things in space and time. Equally, the concepts and purely logical objects, such as numbers, that are bound by the necessary laws of logic and arithmetic, are not visible in nature itself, but are intelligible to our reason alone. Frege makes this clear in a passage that is about the laws of number but that applies to the laws of logic more generally:

> in the external world, in the totality of the spatial, there are no concepts, no properties of concepts, no numbers. The laws of number are thus not really applicable to external things: they are not laws of nature. But they are certainly applicable to judgements that are made about things in the external world: they are laws of the laws of nature. They do not assert a connection between natural phenomena, but a connection between judgements; and the latter include the laws of nature.
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> —GA 119 [§ 87]

In the essay, "On Concept and Object" (1892), Frege remarks that the concepts under which an object falls are the "properties" of that object; so concepts are in nature to the extent that properties belong to natural objects (FR 189-90 / ÜBG 56; see also GA 86 [§ 53]). Concepts, however, are not present in nature *as concepts*. Numbers, in Frege's view, are not found in nature either: they are purely ideal objects that do not exist in space and time and are not given to the senses.[39] Indeed, as Frege maintains in the *Foundations*, "a statement of number contains an assertion about a concept" rather than about actual things. To say that "the Kaiser's carriage is drawn by four horses" is thus to attach the number four – which is itself a logical object – to the *concept* "horse that draws the Kaiser's carriage", rather than to a set of four-legged objects in space (GA 81 [§§ 46]). (We will consider this claim in more detail in chapter 5 below.) Yet even though concepts and numbers do not belong directly to nature, the laws of logic and arithmetic are *objective* because they are not created by thought but have to be "recognized" (*erkannt*) and "grasped" (*ergriffen*) by it (GA 57 [§ 26]). They have their own independent identity that our reason has to discover, and their being *true* does not depend on their being thought by us (see GA 20 [Intro.]).[40]

Like Frege, Hegel also regards the logical and conceptual as objective, as can be seen from a remark about Kant in an addition to the *Encyclopaedia Logic*: "Kant called what conforms to thought (the universal and the necessary) *objective*; and he was certainly quite right to do this" (EL 83 / 115 [§ 41 A2]). As this quotation makes clear, Kant's contribution, in Hegel's view, was specifically to identify conceptual *necessity* with objectivity: so what is objective is independent of our subjective ideas because it is that to which our thought *must* conform. Hegel goes beyond Kant, however, in considering thought also to be objective in another sense.

For Kant, as we saw in volume 1, the necessary categories and principles of the understanding structure the world of human experience but yield no knowledge of things in themselves (though they can be used to think the latter) (see 1: 27, 33, 316). For Hegel, by contrast, categories not only give form to our experience, but also are inherent in things, in *being*, as such; they are thus objective in a stronger sense than Kant acknowledges. As Hegel says (or is reported to have said), "the true objectivity of thinking consists in this: that thoughts are not merely our thoughts, but at the same time the *In-itself* of things and of whatever else is objective" (EL 83 / 116 [§ 41 A2]).

At the end of the *Logic*, being – with all the categories that belong to it – eventually proves to be *nature*. As Moore rightly points out, therefore, for Hegel, in contrast to Frege, "the stuff of thinking" "constitutes the stuff of nature" itself.[41] For Frege, the laws of logic are objective and independent but do not govern nature directly; they regulate the judgements we make about nature. For Kant, the categories and necessary principles of the understanding do structure nature itself, but such nature is no more than the world of our

possible experience. For Hegel, the categories and principles of thought also structure nature, but nature is in turn what being itself proves to be. The categories of thought have "objective value and existence", therefore, not only because they are necessary, and in that sense independent of us, but also because they express "the *essentialities of things*", and so "*understanding*" and "*reason*" are "*in the objective world*" – in nature – itself (EL 56 / 81 [§ 24], and SL 30 / LS 35). (Precisely how the unity of thought and being is justified in a presuppositionless logic is explained in volume 1, chapter 5.)

Logic and psychology

The third similarity between Hegel and Frege is that both distinguish clearly between logic and psychology and reject what is often called "psychologism". It is possible, however, that Frege would consider Hegel to have veered too close to psychologism himself. The reason for this is that there is an ambiguity in Hegel's conception of logic that Frege would shy away from. Hegel's project in the *Logic* is "to present the realm of thought philosophically, that is, in its own immanent activity or, what is the same, in its necessary development" (SL 12 / LS 9). His logic thus discloses how thought and its categories *must* be structured, what is logically necessary. As such, however, it is double-edged.

On the one hand, logic *prescribes* how the categories should be conceived, as opposed to how they are often conceived in ordinary life. That is to say, its task is to "clarify" or "purify" (*reinigen*) categories that often "do their work only instinctively" and are thus "variable and mutually confusing" (SL 17 / LS 17). In this respect, Hegel's conception of logic coincides to a significant degree with Frege's view that logic is prescriptive. For Frege, logic tells us not how we do think in everyday situations, but how we *must* think – more specifically, how we must judge and make inferences – "if we are not to fail of the truth" (FR 246 / L 64);[42] and much the same could be said of logic as Hegel conceives it, albeit without the restriction to judgement and inference.

On the other hand, however, Hegel's logic is not merely prescriptive, because it discloses what thought – and being – must *be* and so *actually are*. It reveals "the pure concept, which is the innermost nature of objects, their simple life pulse, as well as of the subjective thinking of them", that is, "the *logical* nature that animates spirit, that moves and works in it" (SL 17 / LS 16). This is not to say that nature and spirit are the product of dialectical reason alone: for Hegel, what is rational is, indeed, actual, and what is actual is rational, but it is also the case that "what there is [*Dasein*] is partly *appearance*" – that is, contingency – "and only partly actuality" (EL 29 / 47-8 [§ 6 R]). Concrete existence thus deviates in many ways from what is rational; Hegel is as aware of that as anyone. In his view, however, it deviates not merely from what should be, but from the rationality that must be and is actually at work in being.

The *Logic* thus not only discloses how the categories of thought and being should be conceived, but it reveals thereby the dialectical categories that actually structure nature and guide our lives and thinking. Such categories, however, do not govern nature and human activity in every respect (since they are often entangled with contingencies), and they are often only "instinctively and unconsciously" at work within us (or conceived by us in an abstract manner) (SL 19 / LS 19). For Hegel, therefore, we are often subject to a rational dialectic that we do not properly understand – which is precisely why logic has to teach us how categories should in truth be conceived. Hegel also contends that the categories immanent in nature and in our life and thought take time to become explicit objects of consciousness. Our thought thus *comes* to be guided more explicitly by the categories of reason as we grow from children into adults and then acquire greater self-consciousness through further education and reflection (including philosophical reflection); and in history, too, different peoples are animated by categories to different degrees and sometimes even by different, more or less advanced, categories (see 1: 4–5).

Frege, of course, is well aware that "thinking, as it actually takes place, is not always in agreement with the laws of logic any more than people's actual behaviour is always in agreement with the moral law" (FR 247 / L 64). He does not believe, however, that in such cases logic illuminates the concepts and principles that are instinctively or unconsciously active in our thinking. The laws of logic, for Frege, are *merely* "prescriptions for making judgements": they tell us how we should think, but they do not tell us how, often unconsciously, we do actually think (except, of course, when we actually think as we should) (FR 246 / L 64).[43] If the laws of logic were supposed to disclose how we actually think, Frege maintains, "we should have to assign them to psychology", and for him psychology has no place at all in logic (or mathematics) (FR 247, 250 / L 65, 69, and GA 19 [Intro.]).[44] Since, therefore, Hegel's logic discloses a rationality that is actually at work in our thought, one can imagine that Frege would accuse him of conflating logic (which is prescriptive) and psychology (which is descriptive). From Hegel's perspective, however, there is no conflation at all. The point of speculative logic is to discover which categories and other logical forms, if any, are derivable from pure thought and pure being themselves; and those that are so derived prove thereby to have a necessary structure to which thought must conform if it is to grasp the truth, but also to be inherent *in* thought and being. As such, they must be at work in all human thought and activity and throughout nature (even if they are often implicit rather than explicit, and not every category is at work everywhere and at every time in the same way).

Hegel and Frege thus have subtly different conceptions of the task and nature of logic. Nonetheless, each insists that logic should be kept quite distinct from psychology *as he conceives it*. In the *Logic*, for example, Hegel states that "the

additions of psychological, pedagogical, and even physiological material which logic was at one time given, have subsequently been recognized almost universally as disfigurations [*Verunstaltungen*]" (SL 31 / LS 36). This is partly because the observations, laws and rules that have been imported into logic from psychology and pedagogy have been so "shallow and trivial": consider, for example, the platitudinous rule "that one should think through and examine what one reads in books or hears by word of mouth". Yet Hegel also draws a line between psychology and logic (and, indeed, speculative philosophy as a whole) for another reason.

In the modern age, he maintains, "after the revival of the sciences, when observation and experience had become the principal foundation for knowledge of concrete reality", psychology became a more empirical, and less rationalist and metaphysical, discipline (EPM 4 / 11 [§ 378]). Hegel welcomes this modern turn to the empirical; indeed, he describes empiricism more broadly as a "great principle" and notes that the "*emergence* and *development* of philosophical science" has empirical physics as its historical "presupposition and condition" (EL 77 / 108 [§ 38 R], and EN 6 / 15 [§ 246 R]).[45] He insists, however, that an empirical discipline, such as modern psychology, cannot provide a systematic *foundation* for logic or speculative philosophy more broadly. This is because "empirical psychology takes not only the spirit [*Geist*] in general, but also the particular faculties into which it analyses it, from representation as givens [*als gegebene*], without deriving these particularities from the concept of spirit and so proving the necessity that in spirit there are just these faculties and no others" (EPM 5 / 12 [§ 378 A]). It is this lack of necessity – which stems from its reliance on presuppositions or "givens" – that renders empirical psychology unfit to ground, or even to be part of, logic. Logic, for Hegel, discloses how we *must* think – how to conceive the categories properly – and how we *do* think insofar as our thought is determined, more or less explicitly, by the necessities of reason, rather than other contingent factors. This, however, does not introduce psychology into logic, because modern psychology, for Hegel, is an empirical, observational discipline: it observes *only* how we do think, with all its manifold contingencies, and it says and can say nothing about how we must think. Logic and psychology, as Hegel conceives them, are thus quite distinct disciplines and should not be confused with one another.[46]

In the *Foundations* Frege also insists that "there must be a sharp separation of the psychological from the logical" (GA 23 [Intro.]); and in his 1897 manuscript on logic he echoes Hegel's claim that introducing psychology into logic "disfigures" the latter: as Frege writes, "what is referred to as a deepening of logic by psychology is nothing but a falsification [*Verfälschung*] of it by psychology" (FR 243 / L 59). As we have seen, however, Frege conceives of logic as more narrowly prescriptive or normative than Hegel does. Logic tells us how we should think, but it does not tell us how we actually think, any more

than it determines occurrences in nature. It falls to psychology to describe "thinking, as it actually takes place" (*das wirkliche Denken*) (FR 247 / L 64). For Frege, therefore, psychology should be kept outside logic, because there is a clear distinction, as regards thought, between what should be, on the one hand, and what is and occurs, on the other.[47] Hegel, however, does not take this distinction to be as sharp as Frege does: logical necessity is inherent in being and in thought themselves, and so determines, not only how we should think, but also – though by no means exhaustively – what occurs in nature and how we actually think. In Hegel's view, therefore, empirical psychology has no place in logic, not just because it is not normative, but because it fails to disclose what thought *must actually* be.

Yet one might object that Frege does not in fact consider logic to be *purely* normative, and that the difference between him and Hegel is thus not quite as clear as I claim in this section. Weiner, for example, points out that Frege's normative laws of logic are also descriptive in a certain sense: for they determine what logical objects must *be* and how they must relate to one another.[48] This objection is, I think, a fair one. The realm of the "logical", as we saw above, is held by Frege to be objective, and for him this means that objects, such as numbers, have an identity of their own that we do not create but have to recognize and "grasp" (*ergreifen*) (GA 57 [§ 26]). The way in which we discover what such objects are is not by empirical observation but by understanding how we must, or should, think about them; normative logic, which determines how we *should* think, thus discloses what there *is* (at least in the sphere of the logical). The law of identity, for example, determines that every logical object – indeed every natural object considered, logically, as an "object" – not only should be, but *is*, identical to itself; and since this is objectively the case, that law is "not a law that can be violated – we need not fear that one day someone will discover an object that is *not* identical to itself".[49] In this sense, Frege's logic, somewhat like Hegel's, lays down what objects must *be*, albeit in the logical space disclosed by thought rather than in visible nature itself – though it is hard to imagine Frege welcoming objects that are not self-identical into visible nature either, in spite of the fact that logical laws are not supposed to govern nature as such (see GA 119 [§ 87]).

Yet Frege's position remains distinct from Hegel's: for Frege does not claim that finite beings must always, and do always, *think* in accordance with the laws of logic, as he conceives them, whereas Hegel does claim that thought must always be, and is always, structured by categories, even if for the most part "instinctively and unconsciously" (and not every category is at work in all thought in the same way) (SL 19 / LS 19). Frege makes his position clear in the *Basic Laws*. He notes that, while there may be "external circumstances" that prevent us from rejecting the law of identity, there is no *logical* reason why we cannot do so; logic determines merely that we should not do so. Furthermore,

he writes, even if, as a matter of fact, it is impossible for *us* to reject this law, this "does not prevent us from supposing that there are beings who do reject it" (FR 204 / GGA xvii). In Frege's view, therefore, by prescribing how one should think, the laws of logic do not describe how every finite being must and does think, even if they do describe what logical objects must be. In that sense, in relation to the activity of thinking, those laws – in contrast to Hegel's categories – remain purely normative.

Hegel, of course, proves that Frege is right to acknowledge that there may be "beings" who reject the law of identity, for he rejects it himself – or, rather, he both rejects and retains it at the same time by giving it merely partial legitimacy. The category of "identity" as such does not emerge in Hegel's *Logic* until the doctrine of essence, but it is prefigured in the thought that every "something" (*Etwas*) is a *self-relating* reality. Hegel shows, however, that "something relates itself [. . .] *through itself* to the other, because otherness is posited in it as its own moment" (SL 97 / LS 121). Every something is thus not just itself but also other than itself, because it is intrinsically other-related and intrinsically open to being determined by other things. It is what *it* is, yet also *not* just what it is but what other things make of it. To put the point differently, every object is identical to itself but also in some respect "unequal to itself" (*sich selbst ungleich*) – a fact that, as we shall see, fatally undermines Frege's logical definition of "zero" and, indeed, brings to a halt the whole process through which he proposes to define the other numbers.[50] (In the doctrine of essence, Hegel examines "identity" directly and confirms that it is inseparable from difference and non-identity; indeed, he writes, "identity is thus *in itself* [*an ihr selbst*] absolute non-identity" [SL 357 / LW 29].[51] Everything that has an identity, therefore, also differs from itself and is not just itself; in other words, "all things are in themselves [*an sich selbst*] contradictory" [SL 381 / LW 59]).[52]

For Frege, finite beings will not always think in the way that logic (as he conceives it) requires them to, and Hegel proves Frege's point. For Hegel, by contrast, human thought and action must always be, and are always, guided by the categories (as he conceives them), even though not always and everywhere by the same ones and often implicitly rather than explicitly. This difference between the two philosophers is a subtle one, but it is not without significance.

Proof

The fourth similarity between Hegel and Frege lies in the fact that both stress the need for strict *proof* in philosophy and logic. In a preface to the *Encyclopaedia Logic*, Hegel states that "proof" (*Beweis*) is "indispensable for a scientific philosophy" and consists in the "systematic derivation" of ideas (EL 1 / 11 [Preface to 1st edn]); and later in the text he equates such proof with recognizing the *necessity* of a logical content:

> If thinking has to be able to prove anything at all, if logic must require that *proofs* are given, and if it wants to teach us how to prove [something], then it must above all be capable of proving its very own peculiar content, and able to gain insight into the necessity of this content.
>
> —EL 84 / 117 [§ 42 R]

Furthermore, as he makes clear in the *Logic*, Hegel thinks that proof in philosophy should be free of logical gaps.[53]

Now, as we saw in volume 1, speculative logic must begin with indeterminate being and so have no determinate presuppositions. Among the reasons why logic should be systematically presuppositionless in this way are (a) the fact that it must be radically self-critical and so take nothing for granted, nothing on authority, at the start; and (b) the fact that the very idea of a beginning requires that we have not already begun, and thus that we start not with "a first *and* an other" (which would imply that an advance, and so a beginning, has already been made), but with "something unanalyzable, taken in its simple, unfilled immediacy; and therefore *as being*, as complete emptiness" (SL 52 / LS 65).[54]

A further reason for beginning with pure, indeterminate being is, however, this: to avoid starting from something that requires proof but that cannot be proven. In logic, as Hegel conceives it, every determinate content must be proven by being derived systematically from the previous content. The starting point of such logic, however, cannot be so derived – and so cannot be proven – precisely because it is the starting point. Yet if that starting point is something more complex than pure being, it will involve the connection of one element with another, and this connection will *require proof* – proof that, by definition, cannot be given: as Hegel writes, "what is lacking if we make something concrete the beginning is the proof [*Beweis*] which the combination of the determinations contained in it requires" (SL 55 / LS 68). To avoid being in this situation, therefore, logic must begin with that which contains no conjunction of determinations in need of proof, namely pure being. So, for Hegel, the demand that logic prove its content does not mean that it must prove the very starting point of its proof; but it means that such a starting point should not be in *need* of the proof it can never have, and that in turn means that logic must start from utterly indeterminate being.

In Hegel's view, speculative logic is committed to taking nothing for granted and to proving its content; this is the historical presupposition on which such logic rests. The starting point of such logic, however, cannot itself be proven. Accordingly, this logic must begin with that which requires no proof, and so must begin with simple being. That is to say, it must be systematically presuppositionless in the sense outlined in volume 1.[55]

Hegel does, of course, say that the standpoint – and so, by implication, the starting point – of speculative logic is *proven* by phenomenology. We can begin logic with the simple "resolve" to think purely (SL 48 / LS 58), but the "concept" of logic – that is, the unity of thought and being that forms its element – is justified (in the eyes of non-philosophical consciousness) by phenomenology and receives its proof from the latter (SL 28, 38-9, 46-7 / LS 32, 46, 57). Nonetheless, when we begin logic itself, we have to regard that beginning precisely as a new *beginning* and so must abstract from the fact that it is the result of, and presupposes, a prior science. We have thus to treat that beginning as being without proof and as requiring no proof. This means in turn that we have to take it in its pure immediacy as sheer *being*, "*just because* it is here as the beginning" (SL 50 / LS 61). The beginning of logic will be the point from which whatever emerges – if there is anything – will be derived and thereby proven; but, as the beginning, it must not require proof itself and so must simply be indeterminate being (see also 1: 123-6).

For Hegel, speculative logic is the science that emerges when one takes the demand for proof seriously and refuses to allow any assumptions that require proof to remain unproven. Understanding (*Verstand*), however, does not reject unproven assumptions in such a rigorous way. Consequently, Hegel claims, "proof, as it is envisaged by the understanding, is the dependence of one determination on another", and so "in this sort of proof something fixed is presupposed and from it something else follows". Thus, "what is exhibited here is the dependence of a determination upon a presupposition", not a process of proof that starts from indeterminacy and in that sense is presuppositionless (EL 75 / 105 [§ 36 A]). Hegel finds such "uncritical understanding" to be at work outside philosophy, but also within it, for example, in formal logic as well as in the mediaeval proofs of God's existence.[56] He would also have found it to be at work in the thought of Frege.

In Frege's view, as in Hegel's, "science demands that we *prove* [*beweisen*] whatever is susceptible of proof" (FR 310 / LM 94, emphasis added). Frege notes in the preface to his *Begriffsschrift* that proof "can be given purely logically" or can be "grounded on empirical facts", and he contends that "the firmest proof is obviously the purely logical". Such logical proof, he tells us, "depend[s] on the chain of inference being free of gaps" (FR 48 / BS ix-x [Preface]) – as is also true for Hegel – and it is this that he aims to provide for *arithmetic*. Philosophy's aim, for Frege, is thus to prove the truths of arithmetic by deriving them without gaps from "general laws of logic and definitions" and thereby to show such truths to be analytic. More precisely, the task of the *Foundations* is to show that arithmetic is "probably" analytic in this sense, and that of the *Basic Laws* is then to provide the "gapless" proof that it is indeed so.[57] Geometry, by contrast, is synthetic for Frege, since its "general propositions"

cannot be proven by logic and definitions alone, but must be "won from intuition" (though, as is the case for Kant, such intuition is a priori, rather than empirical, so the truths of geometry are themselves synthetic a priori, rather than a posteriori) (GA 42-4, 121 [§§ 13-14, 89]).[58]

Weiner unwittingly suggests a parallel between Frege and Hegel when she writes that Frege seeks to provide "*presuppositionless* proofs of the basic truths of arithmetic".[59] Fregean proof is, however, significantly different from Hegelian proof, for it does not begin from an utterly indeterminate category, but proceeds from "primitive truths" that are deemed to be self-evident and thus presupposed without further reflection. Fregean proof is meant to have no *hidden* presuppositions (and in that sense to be "presuppositionless"); nonetheless, it rests on what are clearly determinate *presuppositions* – presuppositions that, in Hegel's view, require proof but are not supplied with any.

A main aim of Frege's "concept-script" is to present logical proofs in a way that reveals "every presupposition that tends to slip in unnoticed" and so allows it to be removed or rendered explicit (FR 49 / BS x [Preface]).[60] Moreover, like Hegel, Frege frequently criticizes others for basing their arguments on hidden prejudices or presuppositions.[61] Yet Frege's own project of showing arithmetic to be analytic involves "finding a proof and following it back to the *primitive truths* [*Urwahrheiten*]" (GA 27 [§ 3], emphasis added) – truths that, in his view, are the necessary, and unprovable, *presuppositions* of all logical proof. As Dummett notes, "Frege unwaveringly believed that any deductive proof must have a starting point in the form of initial premisses. A complete justification must therefore derive from premisses of which no further justification is possible".[62]

Such premises, which Frege takes to be true and self-evident, are set out in part 2 of the *Begriffsschrift* and, as we saw above (2: 40-1), they include the law of contradiction "if a, then not not a" (axiom 41) and the law of identity "a = a" (axiom 54). These two laws, in Hegel's view, are very far from unprovable or primitive, and in the doctrine of essence (which follows the doctrine of being in the *Logic*) they are derived from what Hegel calls "reflexion" (see SL 354-5 / LW 25). They thus demand proof and are given it in Hegel's *Logic* (or, to be more precise, they are given *limited* justification there, since things prove to be both identical and not identical to themselves). In Frege's logic, however, they are simply assumed to be true, and so presupposed, without further ado, and so they lack the proof that Hegel thinks they should have. From Hegel's perspective, therefore, Fregean logic is a version of the logic of understanding in which "what is exhibited" is simply "the dependence of a determination upon a presupposition" (EL 75 / 105 [§ 36 A]). Hegel's own logic, by contrast, starts from indeterminate being, which contains nothing that requires derivation or proof, and so is systematically presuppositionless. Hegel is thus more rigorous than Frege in allowing nothing in logic to remain unproven when it requires proof.

Logic and truth

The fifth similarity between Hegel and Frege is that both think (in Frege's words) that "truth is the goal of logic" (FR 178 /ASB 32). In Kant's view, general logic tells us how we are permitted to think; or, as he puts it, it "contains the absolutely necessary rules of thinking, without which no use of the understanding takes place" (CPR B 76). These rules, Kant maintains, provide "criteria of truth", since "that which contradicts these [rules] is false" (CPR B 84). Yet they constitute no more than a "negative touchstone of truth", since they tell us which judgements cannot be true, but do not by themselves yield positive insights into the nature of objects. As Kant writes, "nobody can dare to judge of objects and to assert anything about them merely with logic" (CPR B 85). Transcendental logic, on the other hand, lays down, not just the rules governing thought as such, but the more specific "rules of the pure thinking of an object" (CPR B 80); that is, it sets out the epistemic conditions under which alone something can be an *object* for a judging subject. Such logic does tell us about objects, therefore – albeit only about objects of possible experience, and only about their general structure – and so is "a logic of truth" (CPR B 87).

In the *Logic* Hegel acknowledges a debt to Kant's idea of transcendental logic and, indeed, states that what he calls "objective logic" – namely, the doctrines of being and essence – "would correspond in part to what for him [Kant] is *transcendental logic*" (SL 40/ LS 48). In Hegel's view, however, objective and subjective logic, which together make up the whole of speculative logic, do not merely tell us about objects of experience, but they disclose the structure of being itself – being that, later in Hegel's system, will prove to encompass both nature and the human spirit. Speculative logic is thus a logic of truth in the strongest sense: it discloses what there *is*, what there must *be*, not just how objects of cognition must be thought, or just how we should think if we wish to avoid contradicting ourselves. "Logic", Hegel writes, "is to be understood as the system of pure reason, as the realm of pure thought". The necessary structure of thought, however, is also the necessary structure of being. The realm of pure thought disclosed by speculative logic is thus at the same time "*truth as it is without veil in and for itself*" (SL 29 / LS 33).

For Frege, too, "logic is not concerned with how thoughts, regardless of truth-value, follow from thoughts", but (as noted above) "truth is the goal of logic" (FR 178 /ASB 32). More specifically, logic yields positive truths about logical objects (including numbers) and the relations between them – about what such objects should be and what they are. As already noted, however, logic, for Frege, does not tell us directly about being in the form of *nature*: it governs our judgements about nature, rather than nature itself. It tells us that our judgements should obey the laws of logic and of number, but it does not maintain that such laws are "applicable to external things" (GA 119 [§87]). In

contrast to Hegel, therefore, Frege does not take logic to coincide completely with ontology in the strong – pre-Kantian – sense (though, as I remarked above, it is hard to believe that Frege would not consider natural things, as well as logical objects, to be subject to the law of identity).

Frege's logic, of course, also differs from Hegel's logic in another respect: for Frege's is a species of *formal* logic – that rests on the law of non-contradiction – whereas Hegel's is a speculative, *dialectical* logic that does not adhere strictly to formal logical principles.

Logic as "analytic"

The last similarity between Hegel and Frege to which I wish to draw attention is connected to the previous one. This consists in the fact that both regard logic as "analytic" in some sense but consider it nonetheless to yield insights that are new and informative. In Weiner's words, they regard "reason itself as a source of knowledge".[63]

In the first *Critique* Kant maintains that analytic judgements add nothing through the predicate to the concept of the subject, but merely set out "its component concepts, which were already thought in it" (CPR B 11). Such judgements thus tell us nothing new about the subject and so are merely "judgments of clarification". Synthetic judgements, by contrast, "add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis". They do, therefore, tell us something new about the subject and so are "judgments of amplification". Frege rejects Kant's claim that in an analytic judgement the predicate is contained in the subject, because he rejects the idea that, logically, judgements connect a subject-concept and a predicate-concept at all. Yet he takes up and extends Kant's associated idea that an analytic judgement is justified by a principle of logic, namely, the principle of contradiction. For Frege, analytic truths are thus simply those that are derived from, and justified by, general logical laws and definitions, rather than experience or a priori intuition (GA 27 [§ 3]).[64] Since logic itself derives further laws in a "gapless" manner from the basic laws with which it starts, it is obviously an analytic discipline.[65]

In contrast to Kant, however, Frege claims that the basic laws of *arithmetic*, as well as mathematical equations, are also analytic judgements and that arithmetic is thus "a further developed logic" (GA 29, 119 [§§ 4, 87]). He notes, too, that the range of arithmetical laws and truths increased in the nineteenth century with what Weiner calls the "arithmetization of analysis".[66] Since these new arithmetical truths are themselves analytic and so derivable from logical laws, analytic logic in turn must be capable of yielding new insights. For Frege, therefore, Kant's contention that analytic judgements are merely "judgments of clarification" cannot be right. As he writes in the *Foundations*,

"in view of the powerful development of arithmetical doctrines and of their manifold applications, the widespread denigration of analytic judgements and the fairy tale that pure logic is unfruitful cannot be sustained" (GA 47 [§ 17]). Pure analytic logic is as much a source of new knowledge as is experience or a priori intuition.

Like Frege, and unlike Kant, Hegel understands mathematical equations, such as 7 + 5 = 12, to be *analytic* truths (though, as we saw in the last chapter, what he means by "analytic" in this case is not exactly – and in one respect, not at all – what Kant means).[67] Like Frege, Hegel also takes such "analytic" equations to be derivable through logic. In Hegel's view, however, they are derived not from presupposed logical truths (as Frege contends), but from the logical structure of number, as it emerges in the course of speculative logic.

Like Frege, Hegel also takes pure *logic* itself to be analytic, at least to a certain extent. In this case, however, Hegel's understanding of the term "analytic" comes much closer to Kant's conception (so, for him, logic and arithmetic are not analytic in the same way). Hegel describes the course of speculative logic as "analytic" while commenting on the opening dialectic of being and nothing in the *Encyclopaedia Logic*. Referring directly to pure being and pure nothing, he states that "the deduction of their unity" – in becoming – "is to this extent entirely *analytic*; just as, quite generally, the whole course of philosophising, being methodical, i.e. *necessary*, is nothing else but the mere *positing* of what is already contained [*enthalten*] in a concept" (EL 141 / 188 [§ 88 R]).

Hegel's formulation is clearly reminiscent of Kant's conception of the analytic "containment" of one concept in another. Yet one must exercise caution here, for, strictly speaking, neither pure being, nor pure nothing can "contain" anything: as Hegel insists, pure being "has no difference within it", and pure nothing is the "complete absence of determination and content" (SL 59 / LS 71-2). What makes the first stage of speculative logic analytic, therefore, is not the "containment" of one concept in another, but rather the process of "positing", or rendering explicit, what a concept *itself* is "implicitly". Becoming is "analytically" connected to being, because it is simply the vanishing that being itself, in its utter indeterminacy, proves explicitly to be; similarly, becoming is the vanishing that nothing, in its sheer immediacy, proves to be (see 1: 76, 147). Speculative logic proceeds in the same way for the rest of its course, even though the categories become more complex. New categories are always analytically connected to the ones that precede them, since they owe their identity solely to the latter. Yet they are not already "contained" in those preceding categories; rather, a new category *emerges* as thought renders explicit what a specific category is implicitly. For Hegel, therefore, not only are logic and arithmetic analytic in different ways, but neither is analytic in exactly Kant's sense.

What we have just set out, however, is not the whole story: for Hegel notes that the unity of being and nothing in becoming is equally their absolute difference: "the one is *not* what the other is" (EL 141 / 188 [§ 88 R]). After all, being proves to be becoming by vanishing into *nothing* and thereby into its opposite; being and nothing are thus united in becoming *as opposites*. In this sense, becoming can be understood as the "synthesis" of distinct categories. This moment of synthesis is, however, inseparable from the analytic character of logic: becoming is analytically connected to both being and nothing, because it is simply what each *itself* proves to be; but it is equally the synthesis of the two, because each proves to be becoming by vanishing into its opposite. In other words, becoming is the synthesis that emerges analytically at the start of speculative logic.[68]

Hegel's logic is thus both different from and similar in certain ways to Frege's. Frege's logic is analytic because it simply works out – or renders explicit – what follows from basic laws, such as those of identity and contradiction. The truths of arithmetic are also analytic, for Frege, because they arise from rendering explicit – using such general laws of logic – what is implicit in the definitions of number and of the individual numbers. These arithmetic truths are said by Frege to be "contained" (*enthalten*) in those definitions. Yet they are not contained in a Kantian sense, for, as Frege insists, they are contained "like a plant in a seed, not like a beam in a house": they are something *new* that emerges from the definitions and logical laws (GA 121 [§ 88]). As we noted above, purely analytic logic is thus, for Frege, a source of new knowledge.

Hegel also thinks of the categories in speculative philosophy as connected to one another "organically", rather than mechanically; and he compares (albeit obliquely) the "progressive development of truth" in the history of philosophy to the bursting forth of a blossom from a bud (PS 2, 20, 34 / 4, 27, 42). Hegel's logic differs from Frege's, however, because in it categories turn, of their own accord, *into their opposites*: being vanishes into nothing, something proves to be an other, and so on. This is the *dialectical* moment that Hegel discerns at the heart of being and thought, and that Frege, wedded as he is to identity and non-contradiction, would regard as "madness" (*Verrücktheit*) (FR 203 / GGA xvi).

Such dialectic, for Hegel, is not at all mad, but is rational and necessary. Furthermore, it is what actually produces novelty in logic and in philosophy more generally: such novelty arises because one category, in being what it is, passes over into what it is not. For Hegel, therefore, it is not the analytic character of logic alone that yields new truths. New truths emerge because the analytic process of rendering explicit what is implicit in a category coincides with that category's *dialectical* transition into its opposite (or, at least, with its mutation into a different category). For Frege, by contrast, analytic truths *without* dialectic lead to new insights.

There are, therefore, significant similarities between Frege and Hegel – similarities that are rarely remarked upon. Yet the two remain very different

thinkers with different aims and methods. The principal difference between the two is that Frege rests his thought, and his account of number, on certain definite *assumptions*, whereas Hegel endeavours to avoid assumptions altogether. In the next chapter we will look more closely at the most important assumptions that guide Frege's thought.

CHAPTER FOUR

Excursus: Frege's Assumptions

FORMAL AND SPECULATIVE LOGIC

Hegel acknowledges that the method he follows in speculative logic, or rather that the system of logic itself follows, is "capable of greater perfection, of much elaboration in detail". Yet he also insists that "it is the one and only true method" (SL 33 / LS 39).[1] This is because it is the method that belongs to and is immanent in "its object and content", namely being (and subsequent categories) – "*the dialectic which it possesses within itself*". Such immanence is in turn secured by holding all assumptions about being at bay and letting that "content" unfold through its own dynamic alone. Speculative logic thus discloses the true nature of being (and thought) because it is systematically presuppositionless: it lets the matter at hand "have free play" (*gewähren*) and allows no external presuppositions to determine the course of its logical development.[2]

For Hegel, therefore, speculative logic, rather than Kant's transcendental logic, is the genuine "logic of truth". Hegel is well aware, however, that besides speculative (and transcendental) logic, there is also another form of logic: *formal* logic. This logic, which in the philosophical tradition is usually equated with logic *tout court*, was founded – or at least first clarified and systematized – by Aristotle and, Hegel claims, "has made no progress since Aristotle's time", though it was extended by the Stoics and the mediaeval Scholastics (VGPW 2: 229, 237, 273 ff.). It is described as "formal" because it lays down what many regard as the "formal conditions of true cognition", or the rules of correct thinking, whatever the content may be (SL 24 / LS 26). Speculative logic, as just noted, discloses the logic inherent in its peculiar content – pure being – but formal logic sets out rules that are held to govern the thought of *any* content

and so to be peculiar to none. It is thus logic understood as the "science of thought in general".

Speculative and formal logic also differ in another respect: whereas speculative logic begins from no systematic presuppositions and so starts with (the thought of) pure being, formal logic starts from very definite presuppositions. These (according to Hegel) include the categories and forms of syllogism that are set out by Aristotle, and, above all, *identity* as conceived by the understanding, or the principle "that nothing should contradict itself" (VGPW 2: 240; see also SL 18 / LS 18).

In contrast to formal logic, speculative logic (in the doctrine of essence) *derives* the principle of identity from the structure of "reflexion" which is in turn ultimately derived from pure being.[3] In that sense, speculative logic justifies the main principle on which formal logic rests. At the same time, however, speculative logic shows identity to be inseparable from non-identity and contradiction, and so to be only part of the truth: things do have an identity of their own, yet they are not merely self-identical but are in a certain respect standing contradictions, too. In this sense, the main principle on which formal logic rests is shown to be an abstraction or simplification (though not for that reason a fiction) and so is given only partial justification. Formal logic itself thus proves to have only partial, limited validity: it is valid for what Hegel calls "finite thinking" which is guided by one-sided abstractions and so only ever sees one side of the truth at a time; that is, it is valid for the "abstract understanding" (EL 52 / 75 [§ 20 A], and VGPW 2: 237, 240). It is not valid, however, for thought – whether in art, religion or philosophy – that is sensitive to the many-sided, dialectical complexity of things.[4]

Speculative logic also derives, and justifies, the various forms of judgement and syllogistic inference that are presupposed in formal logic (see SL 557 ff., 590 ff. / LB 66 ff., 106 ff.). Yet by showing them to be dialectically connected to one another, speculative logic shows that each on its own is one-sided, or, as Hegel puts it, that "individually they have no truth" (VGPW 2: 239). Formal logic, which takes one form of syllogism at a time as the foundation of a particular pattern of reasoning, can thus, once again, only disclose one side of the truth.

In contrast to formal logic, speculative philosophy conceives of syllogisms as an integrated "totality" (VGPW 2: 239).[5] Furthermore – again in contrast to formal logic – it holds that the "syllogistic form is a general form of all things", not just the form of our thinking *about* them: the syllogism, like the judgement and the concept, is an ontological, as well as a logical, form.[6] For speculative philosophy, therefore, certain objects in nature and in the human world are themselves structured by a unity of different syllogisms: "in the practical sphere, for instance, the state is a system of three syllogisms just like the solar system" (EL 276 / 356 [§ 198 R]). Such ontological structures, in Hegel's view, are necessarily beyond the grasp of merely formal logic.

Note, however, that the development of speculative logic itself is not governed by syllogisms, even when they are taken as an interconnected system. Such logic discloses the dialectic inherent in one category, or form of syllogism, that connects it to another, but that dialectic does not itself proceed in accordance with any syllogism or set of syllogisms (nor can it be expressed in a set of simple judgements).[7] In a syllogism (conceived as a form of thought, rather than an ontological structure) a conclusion follows from a major and a minor premise; to present dialectic in this way, however, is to eliminate altogether its *dynamic* character. Dialectic *can* no doubt be represented syllogistically, but to do so is to fail to capture the movement in which a category *turns into* its opposite (or at least into a different category). It also risks reducing dialectic to a general pattern of thought and so obscuring the fact that it is the specific dynamic generated by a specific category. Dialectic is thus beyond the grasp of formal logic. This is not to deny that the understanding plays a significant role in speculative logic (see 1: 78-85). Such logic, however, and the dialectic it describes are not *governed* by the principles and laws of the understanding as these are set out in formal logic.

Hegel clearly considers formal logic, and the close study of it, to be important and valuable. "There is no doubt" he says, "that working on this formal logic has its use. Through it, as people say, we sharpen our wits; we learn to collect our thoughts, and to abstract". Furthermore, "acquaintance with the forms of finite thinking can be used as a means of training in the empirical sciences, which proceed according to these forms" (EL 52 / 75-6 [§ 20 A]).[8] Yet it is also clear that, for Hegel, formal logic has limited justification and limited application: it lays down rules that govern the one-sided understanding but that do not govern all thought as such. Kant maintains that formal, or "general", logic considers "the form of thinking in general [*des Denkens überhaupt*]" (CPR B 79). In Hegel's view, however, this is not true: formal logic is not *generally* valid for and applicable to all thought, because it does not govern speculative logic. Speculative logic may not, and does not, presuppose the laws of formal logic because, as a fully self-critical discipline, it may not begin with any determinate presuppositions at all: its task is precisely to discover, not to assume in advance, what may be inherent in pure thought (and being). Nor does speculative logic, as it develops, turn out to be governed by the principles of identity and non-contradiction or to proceed syllogistically, for it proves to be dialectical. (Furthermore, it provides only limited justification for those principles and for the individual forms of syllogism.)

Now Frege is clearly a formal logician in Hegel's sense: the laws of identity and non-contradiction, as we have seen, are axioms of his logical system (see 2: 40-1), and the single mode of inference, *modus ponens*, to which he reduces all others, matches what Hegel calls the "hypothetical syllogism" (namely, "if A is, so is B; but A is; therefore B is").[9] Yet Frege also revises formal logic in what

many see as a "revolutionary" manner by, for example, introducing quantifier notation (see FR 1, 10). My concern here, however, is not to compare Frege's and Hegel's conceptions of formal logic as such, or to consider in detail how Hegel might regard Frege's innovations in that area. Such a comparison can be undertaken only after close examination of Hegel's account of judgement and syllogism in the doctrine of the concept, and that falls outside the scope of this study. My concern here is to consider more closely the relation that Frege sees between logic, on the one hand, and *arithmetic* and *number*, on the other, in light of Hegel's understanding of that relation.

As we have seen, Hegel derives number from the nature of quantity, and he derives the latter, via quality, from pure being. Hegel's derivation of number is thus systematically presuppositionless. Furthermore, since he thinks that arithmetic is explained by the distinctive logical structure of number, his account of arithmetic is also presuppositionless, even though it does not belong to the main line of logical development in the *Logic*, but is set out in a remark (see 2: 25 ff.). As is evident from this study and from my other work, I take Hegel's claim to systematic presuppositionlessness seriously; indeed, in my view, what gives Hegel's thought its power and urgency is precisely the fact that it is, at its beginning, *less question-begging* than any other way of thinking.[10] I have also endeavoured to show in this study, and elsewhere, that Hegel introduces no unwarranted assumptions in his unfolding of the thought of pure being, and that, consequently, the categories that emerge in the course of his speculative logic – including that of number – are well-grounded. Readers must judge for themselves whether this is true, but this is the conviction that I bring to the consideration of Frege.

In contrast to Hegel, Frege, as a formal logician, founds his logic on several determinate presuppositions. These include the law of identity, which he takes to be primitive, as well as other axioms set out in his *Begriffsschrift*. Frege also assumes that logical thought involves judgement, inference, negation, the conditional ("if a, then b"), and the universal quantifier ("for any x"), without, as far as I can see, deriving any of them from thought itself (and, of course, by simply assuming that judgement is essential to thought he closely resembles Kant). He also makes other assumptions that we are about to consider and that are of particular relevance to the relation, as he understands it, between logic and arithmetic. In my view, all these assumptions taken together render Frege's logicist account of arithmetic and number much more question-begging than Hegel's, and so undermine the claim that his account might represent "an enormous advance" in rigour over Hegel's (to use Parsons' phrase).[11] This is by no means to deny that Frege's conception of number has been regarded by many as fruitful for the philosophical understanding of arithmetic. The question, however, is whether it is more adequately *justified* than Hegel's, and on my understanding it cannot be because it is more question-begging.

LOGIC AND ARITHMETIC

Frege's first assumption, relevant to the relation between logic and arithmetic, is that "thought is essentially the same everywhere" and that "it is not the case that there are different kinds of laws of thought depending on the object" (GA 17 [Intro.]). For Frege, as for Kant, therefore, all thought is governed by the same laws: those of formal logic.[12] Frege combines this assumption with a second, namely that arithmetic also has universal application. "Arithmetic truths", he writes, "govern the realm of the countable"; but "this is the most comprehensive; for not only does the actual, the intuitable belong to it, but so too does everything conceivable [*alles Denkbare*]" (GA 44 [§14]).[13] Since both logic *and* arithmetic are taken by Frege to apply to everything that can be thought, the following idea occurs to him: might not the laws of arithmetic stand in "the most intimate connection" with the laws of logic? Indeed, might not the former be derivable from the latter and so be analytic?

Frege's twofold assumption that logic and arithmetic both have universal applicability does not itself prove that arithmetic has a purely logical ground: the two disciplines might share the same domain by accident. Yet that assumption provides what Dummett calls a "suasive" reason for believing that arithmetic is analytic.[14] From Hegel's perspective, however, both parts of Frege's assumption lack justification since they are mere *assumptions*. Furthermore, one is clearly mistaken and the other requires a crucial qualification, absent from Frege's account, in order to be tenable.

First, thought, for Hegel, is *not* "essentially the same everywhere", or at least not explicitly so. The same categories are, indeed, at work "instinctively and unconsciously" in much human thought (SL 19 / LS 19), but thought nonetheless also takes explicitly different forms (that in some cases are governed by different categories, such as "concept" or "number").[15] We have already seen that speculative thought, which proves to be dialectical, thereby proves to be different from one-sided understanding, which is governed by formal logic. Hegel also contends, however, that there is a difference between such formal thought, which he understands to be syllogistic, and arithmetical calculation. As he will show in the doctrine of the concept, syllogistic thought involves discerning logical connections between *concepts* and *judgements*: it involves recognizing how one concept "inheres" in or is "subsumed" under another, or (in the hypothetical syllogism) how the conditional "if A is, so is B" as major premise and its antecedent "A is" as minor premise make it necessary to conclude that "B is" (SL 588, 620 / LB 104, 143). Arithmetical calculation, by contrast, is performed with numbers and units that have no conceptual connection with, but are quite external to, one another. As Hegel writes, in such calculation "the understanding does not even have the formal, abstract determinations of the concept before it": numbers do not "inhere" in one another, as the universal inheres in the particular, but they are (and contain) mutually indifferent units

that are combined and separated externally (SL 603 / LB 122-3; see also SL 171 / LS 216-17). Arithmetic, Hegel argues, does employ a form of syllogism and so is not utterly unrelated to other forms of syllogizing. This "mathematical" syllogism states that "if *two things or determinations are equal to a third, then they are equal to each other*" (SL 602 / LB 122). Hegel points out, however, that "the relation of inherence or subsumption of the terms is extinguished [*ausgelöscht*]" in this syllogism. This is because, as just noted, its terms are neither concepts nor judgements, but numbers and units that are wholly *external* to one another. (Such externality also means that arithmetic cannot provide a model for speculative philosophy in which categories turn *into* one another dialectically [see e.g. PS 25-6 / 32-4].)

Now at the stage we have reached in the *Logic*, the syllogism has not yet been derived and so proven to belong to thought. Hegel's remarks on the mathematical syllogism thus cannot themselves justify the claim that arithmetical calculation differs from other thought governed by formal logic. The mathematical syllogism, however, posits an "equality" (*Gleichheit*) or "inequality" (*Ungleichheit*) between its terms because it abstracts from the conceptual relation between them (the relation of universal to particular or individual), as well as from their qualitative difference, and treats them as mere *quantities* (SL 603 / LB 122). The distinctive character of arithmetical calculation is thus grounded ultimately, not in a particular form of syllogism, but in the logical structure of quantity, and more specifically of number; and quantity and number *have* been derived and so proven to be logically necessary at this point in the *Logic*. We are justified by the nature of quantity and number, therefore, in maintaining that arithmetical calculation differs from thought governed by formal logic.

Since, in Hegel's view, numbers and their component units are external and indifferent to one another, arithmetical calculation with them is itself an external, "thoughtless" and "mechanical" operation (SL 181 / LS 230).[16] This is not at all to say that it is easy and demands no intellectual effort; but "the effort consists above all in holding fast what is without concept [*Begriffloses*] and combining it in a non-conceptual [*begrifflos*] manner". It does not require the kind of rational insight that is needed to know whether this concept falls under that one or not, or whether this judgement follows from those. Indeed, Hegel argues, it is precisely because calculation is such "an external and therefore mechanical business" that "it has been possible to construct *machines* that perform arithmetical operations with complete accuracy" (SL 181-2 / LS 230). Hegel clearly considers understanding that is governed by formal logic to be less insightful and less rigorous (because more question-begging) than speculative reason, but he does not deny that it requires conceptual thought, judgement and reasoning; one could not, therefore, construct a machine that is capable of genuine understanding – one that can judge and reason as a human

being can. A *calculating* machine, by contrast, is possible because of the distinctively mechanical character of arithmetical calculation. Accordingly, from Hegel's point of view, Frege is wrong to assert that "thought is essentially the same everywhere", and so wrong to claim that the laws of formal logic are binding on all thought (whether or not we actually abide by them).

Frege decisively rejects the idea that arithmetical calculation consists in what he calls "aggregative, mechanical thought"; indeed, he "doubt[s] that there is any such thought" (GA 17 [Intro.]). In his view, calculation is governed by the same laws, and so is *inferential* in the same way, as all logical thought (see GA 119 [§ 87]). His hostility to the idea that arithmetic is aggregative and mechanical goes hand in hand with his hostility to the thought "that numbers are formed in a particularly mechanical way, as sand, say, is formed from grains of quartz" (GA 17-18 [Intro.]). From a Hegelian perspective, however, Frege dismisses this latter thought because he severs the connection between number and the *quantum* – which does, indeed, contain a "plurality of ones" (*Vielheit der Eins*) (SL 168 / LS 213) – in favour of the connection between number, arithmetic and the laws of formal logic. This brings us to what Hegel would see as the problem with Frege's second assumption, relevant to the relation between arithmetic and logic: namely that *arithmetic*, just like logic, is universally applicable to all that can be grasped by conceptual thought. For Hegel, in contrast to Frege, arithmetic does not apply without further qualification to "everything conceivable", but applies to the latter only insofar as it is (or can be regarded as) *quantitative*. This is because "*arithmetic* treats number and its figures" and a number is itself nothing but a fully determinate quantum (SL 171 / LS 216). The sphere of quantity, rather than the conceivable as such, is thus the domain of arithmetic. This is not to deny that number, as Hegel conceives it, is grounded in logic. It is grounded, however, not, as Frege claims, in formal logic, but in the logic, inherent in being itself, that makes quantity necessary.

Hegel's association of number with the quantitative explains why he thinks that some things lend themselves more to being counted than others. A number, for him, is a fully determinate quantum and, as such, is a collection of self-relating units and a self-relating unit itself. Something is most clearly countable, therefore, if it contains such units or is itself such a unit; so grains of sand, while being more complex than mere separate units, are nonetheless separate units and so can be counted. In Hegel's view, however, the categories of speculative logic are importantly different.

Some of them are, indeed, separate and self-contained: "something" and "other", for example, are explicitly *self-relating* categories, and so Hegel can say of them that they are "two somethings" (SL 97 / LS 121). The "one" (*Eins*) is of course also separate, even more so than "something", and so can be counted (initially, at least) as *one* category on its own; and the many "ones" would be countable, too, if they constituted a determinate plurality. The logical

structure of "something" or the "one" is *not yet* that of the quantum or number itself (which is a quantitative, not a qualitative, one); but their separateness allows them to be regarded *as* quanta and so to be numbered. Other categories are not so clearly separate, but are bound together as sides of a single difference (for example, reality and negation); nonetheless, the fact that they are explicitly different also permits us to think of them as two categories, though with less obvious justification than in the case of something and other. *All* of the categories, however, display a characteristic that makes them logically very different from grains of sand and less amenable to being counted. This is not merely that they are, precisely, logical rather than physical entities; it is that they are irreducibly *dialectical* and pass over into their opposites (or into other categories) through simply being what they are. In the case of the categories, therefore, their separateness, where they exhibit it, is (or proves to be) merely a *moment* of their logical nature; it is not (or does not remain), as in the case of the sand grains, their principal feature. Thus, although the categories – and, indeed, their moments – *can* be separated and counted, a certain violence is done to them in so doing, in a way that it is not done to certain other things.[17]

Indeed, in the second remark after the introduction of number Hegel highlights what he considers to be the "madness" of treating the categories as countable entities. "Thought is set its hardest task", he writes, "when the determinations for the movement of the concept, through which alone it is concept, are denoted by one, two, three, four"; for thought then "moves in the element of its opposite, the element devoid of all relation; [and] its business is the work of madness [*Verrücktheit*]" (SL 180 / LS 228; see SLM 215). Once again, this is not to say that one *cannot* treat the categories as separate, countable elements and deprive them of their distinctive dialectical relation to one another; but it would be irrational to do so, since dialectic belongs to their true nature. Frege, of course, would see madness in dialectic itself, since it violates the principle of non-contradiction (FR 203 / GGA xvi);[18] from a Hegelian point of view, however, Frege exhibits a madness of his own by insisting that *everything* conceivable, including the categories of thought, belongs in the same way to the "realm of the countable" (GA 44 [§14]).

For Frege, arithmetic applies to what is conceivable without qualification. It thus has the same range as formal logic and so is likely to be derivable from it. For Hegel, by contrast, number and, via number, arithmetic are derivable logically from *quantity*, not from formal logic. Arithmetic thus applies to everything conceivable only insofar as it is, or can reasonably be regarded as, quantitative. This means, however, that not everything is amenable to counting and to arithmetic to the same degree or in the same way, because not everything is equally quantitative. Hegel notes in the *Encyclopaedia*, for example, that "quantity plays what we may call a more important role in inorganic nature than in organic" (EL 159-60 / 212 [§ 99 A]); and it is clear from the *Logic* that,

for him, "spiritual" conceptions, such as that of the Trinity in Christianity, are seriously distorted when quantified (SL 180 / LS 228) – though these claims cannot be justified until later in Hegel's system. The greatest irrationality, however, resides in seeking to quantify and enumerate the categories of thought: for in that way their distinctive dialectical dynamic, which *has* already proven itself by this point in the *Logic*, is lost.

So, to recapitulate: Frege is drawn towards incorporating arithmetic into logic because he assumes that both are universally applicable to what is conceivable. As we have just seen, however, Hegel would regard this twofold assumption as problematic. In his view, formal logic – to which Frege remains wedded, despite his innovations – does not govern all thought, for it governs neither speculative logic, nor, indeed, arithmetical calculation, which is essentially mechanical rather than properly inferential. Furthermore, in contrast to both speculative and formal logic, arithmetic applies to everything conceivable only insofar as it is (or can be taken as) *quantitative*, and not everything is quantitative in the same way or to the same extent. Since Frege's assumptions are problematic in this way, they provide no firm ground for believing that arithmetic is derivable from formal logic; and, indeed, Hegel's own speculative logic shows that arithmetic (properly conceived) is not so derivable.

There is a further problem with Frege's association of arithmetic and formal logic, namely that it rests on what looks like a circular argument. Hegel's logic has no particular goal to reach or aim to fulfil: it does not set out to prove that this or that category belongs to thought, or that arithmetic has a particular character. Its aim is simply to think pure being and to discover what, if anything, is made necessary by it. In the process speculative logic discloses that pure being, of its own accord, makes quality and then quantity necessary, and that quantity in turn takes the form of number. Such logic thus discovers the nature of number on the basis of no prior assumptions about the latter or about arithmetic.

By contrast, Frege does have a particular aim to fulfil, namely to demonstrate that arithmetic is analytic and therefore grounded in formal logic. Like Hegel, however, Frege is also interested in logical proof and, indeed, in presupposing as little as possible about thought (though, as a formal logician, he begins with definite assumptions or "primitive truths", rather than indeterminate being) (see 2: 49-52). Given this interest of Frege's, one would expect him to begin by examining (what he takes to be) the laws of logic for their own sake and then to discover through doing so whether or not arithmetic is logical and analytic. Yet he does not proceed in this open-ended way; rather, he revises logic *so that* arithmetic can be derived from it, since he is antecedently convinced that arithmetic should be analytic (since both have universal applicability). Moreover, he draws specifically on *mathematics* and *arithmetic* in order to revise logic: as he freely admits, "arithmetic [. . .] was the starting point of the train of thought

that led me to my *Begriffsschrift*" (FR 51 / BS xiv [Preface]). Frege's conception of the logic that is to ground arithmetic is thus determined (at least in part) *by* the very arithmetic it is meant to ground, and in this respect his argument for the analytic character of arithmetic looks decidedly circular.

It is true that Frege also has other aims in reforming logic. He wishes to free logic from what he sees as the distortions of natural language (FR 50-1 / BS xii-xiii [Preface]);[19] and he seeks to develop a single logic, and system of logical notation, that makes it possible to evaluate "an inference of whatever kind", whether it is an Aristotelian syllogism (based on the relation between concepts) or an inference in propositional logic (based on the relation between propositions) (FR 59 / BS 9 [§ 6]).[20] Hegel unites propositional logic with Aristotelian syllogistic *dialectically* by deriving the hypothetical syllogism, or *modus ponens*, from simpler forms of the syllogism; but he was not concerned specifically to construct a single formal logic that would enable us to evaluate both ordinary syllogisms and propositional inferences together. Frege, by contrast, clearly was.

Yet, by his own admission, Frege's principal reason for reforming logic was to make it possible for it to ground arithmetic. Frege does not *discover* that logic, quite independently of arithmetic, grounds the latter, but he tailors his logic to suit the arithmetic it is meant to ground.

"FUNCTION" AND "ARGUMENT" IN FREGE'S THOUGHT

So how exactly does Frege render formal logic suitable to ground arithmetic? As Michael Beaney puts it, "Frege's crucial move lay in extending the idea of *function-argument* analysis from mathematics to logic" (FR 10, emphasis added). In his essay, "Function and Concept", Frege explains that a mathematical function is the "common element" in expressions such as "$2.1^3 + 1$", "$2.4^3 + 4$" and "$2.5^3 + 5$", and that this common element can be represented as "$2.x^3 + x$" or just as "$2.(\ )^3 + (\ )$". The argument, by contrast, is the element that changes in the first three expressions, namely, 1, 4 or 5, and that is indicated by "x" in the fourth. Since "x" in the fourth expression is a variable standing for any *argument*, it does not, strictly speaking, belong to the function itself. The pure function, as opposed to the argument, is thus most accurately represented by the final expression, "$2.(\ )^3 + (\ )$", in which there are simply empty spaces for an argument to fill (FR 133-4 / FB 5-6).

Two things should be noted about mathematical functions and arguments, as Frege understands them. First, as is clear from the empty spaces it contains, "a function by itself must be called incomplete, in need of supplementation, or unsaturated [*ungesättigt*]"; by contrast, "the argument is a number, a whole complete in itself, as the function is not" (FR 133-4 / FB 5-6). Frege argues

later in "Function and Concept" that functions can themselves be the arguments of "second-level" functions (FR 146 / FB 19). This, however, can be ignored here: for what is important for our purposes is simply the fact that, in order to render logic fit to ground arithmetic, Frege imports into the former the distinction between an incomplete *function* and an *argument*.[21]

Second, the distinction between an argument and a function has nothing to do with that between subject and predicate. An expression, such as "$2.1^3 + 1$", contains no subject or predicate, because it does not express what Frege calls a "judgeable content" – one that can be judged true or false (FR 53, 55 / BS 2, 4, 5 [§§ 2, 4-5]) – and only such a content can possibly be understood to attribute a predicate to a subject.[22] The equation, "7 + 5 = 12", however, does express a judgeable content, so could it not contain a subject and predicate? Kant would say that it does: for he thinks that this equation is a synthetic judgement in which the predicate-concept "12" adds something to the subject-concept "7 + 5" (namely the amount that results from the operation symbolized by "7 + 5") (CPR B 15-16). We saw above, however, that Frege rejects Kant's conception of a mathematical equation (2: 39-40). For him, "the relation of equality, by which I understand complete coincidence, identity, can only be thought of as holding for objects, not concepts" (FR 175 / ASB 28). The equation "7 + 5 = 12" does not, therefore connect a subject-concept and a predicate-concept, but states that the *object* denoted by the sign "7 + 5" is the very same as the *object* denoted by the sign "12" (namely the number 12).[23]

Furthermore, the distinction between the function and the argument in an equation, such as "7 + 5 = 12", is itself independent of the distinction between subject and predicate that someone (like Kant) might want to see in it. Since the function in an expression is the element it has in common with other expressions, it is in fact possible to identify several different functions in the expression "7 + 5 = 12". One such function might be "( ) = 12", which can take the arguments "7 + 5", "6 + 6", "36 ÷ 3", and so on; and in this case, the function would, indeed, resemble, if not exactly coincide with, the "predicate" that Kant sees in the equation. Another function, however, would be "( ) + ( ) = ( )", which could take various different sets of arguments: "7, 5, 12", "1, 2, 3" and so on. In this case, it is clear that that mathematical function has nothing to do with the "predicate" of the expression in Kant's sense, but is simply the incomplete element it shares with other equations. This example shows, therefore, that the argument / function distinction and the subject / predicate distinction are independent of one another.

When Frege imports the former distinction into *logic*, it remains independent of the latter one; indeed, it replaces, and enables him to dispense with, that latter one. Certain sentences may retain a grammatical subject and predicate, but, Frege insists, "the grammatical categories of subject and predicate can have no significance for logic" (FR 242 / L 58). The independence of the

argument-function relation from the subject-predicate relation is made clear by Frege in his *Begriffsschrift*.

Take, for example, the claim that "hydrogen is lighter than carbon dioxide". In this expression, we can consider hydrogen to be the argument – replaceable by, say, oxygen – and the expression "is lighter than carbon dioxide" to be the function that remains invariant when the argument is changed. We can, however, express the same thought, or "judgeable content", by saying that "carbon dioxide is heavier than hydrogen", in which case we can consider "carbon dioxide" to be the replaceable argument and "is heavier than hydrogen" to be the invariant function (FR 65-6 / BS 15-16 [§ 9]). In both cases, the division of the sentence into argument and function, as we have presented it, still coincides to a large degree with its division into subject and predicate: so in the first expression "hydrogen" can be thought as the subject and "being lighter than carbon dioxide" as the predicate. Yet the argument and function in this expression can also be distinguished in a way that does not match the subject-predicate distinction at all: for, as Frege points out, "'the circumstance that hydrogen is lighter than carbon dioxide' can be taken as a function of the two arguments 'hydrogen' and 'carbon dioxide'" (FR 68 / BS 18 [§ 9]). In this case, the function itself must be the expression "( ) is lighter than ( )". On this analysis, Weiner notes, "there is no subject", since "the statement is as much about carbon dioxide as it is about hydrogen";[24] and "is lighter than" is clearly not a conventional predicate-concept that joins with a subject-concept and a copula to form a judgement, but is a two-place function that expresses, or rather is, a relation.[25] Indeed, the function could be considered to be even more abstract, namely "( ) is ( )-er than ( )", in which case it is a three-place function and even less like a conventional predicate.

It is important to note that Frege's introduction of the argument-function distinction into logic allows him to conceive of functions with *two or more* arguments as constituting judgeable contents. This then allows him to conceive of *relations* between two or more arguments as constituents of such contents (see e.g. GA 102-4 [§ 70]). According to Weiner, "Aristotelian logic depends on an understanding of judgments that makes it impossible to recognize complex arguments involving relations as logically valid".[26] This is because the premises of an Aristotelian syllogism are assertions with a single subject – which may be "S", "some S" or "all S" – and a single predicate, not assertions that relate one subject (or object) to another.[27] Frege's revised logic, by contrast, can evaluate inferences in which relations are premises and conclusions. This in turn permits him to derive general truths about "series" or "sequences" (*Reihen*) using logic alone, since sequences are formed by means of two-place relations: for example, by means of the "less than" or "greater than" relation.[28] Furthermore, Frege contends that general truths about sequences will apply to the sequence of natural *numbers*, as much as to other sequences. So if x is less than y, and y is

less than z, then x is less than z, whether x, y and z are the sizes of buildings or numbers; either way, the inference is valid.

We can also derive the principle of "mathematical induction" from a general logical truth about sequences. According to Weiner, mathematical induction licenses us to infer that if a property holds of the first member of a natural number sequence, and if it is *hereditary* in that sequence (that is, if it also holds of the successor of any member of which it holds), then it must hold of every member of the sequence.[29] What justifies this inference, however, is not some special feature of numbers, but a general logical law that governs sequences as such. It is true of *any* sequence, whether its members are people or numbers, that if a property belongs to its first member and is hereditary, it belongs to all its members; this inference, like other inferences about sequences, is an inference "about sequences of any sort of objects".[30] The principle of mathematical induction is thus "demonstrable from logic alone", and once this principle has been secured we have taken the vital first step towards proving that all the truths of arithmetic are analytic: for we prove this latter claim, in Weiner's words, by going on to show "that the conclusions we derive using mathematical induction require only definitions and logical laws".[31]

The details of Frege's derivation of the properties of sequences are not my concern here. My concern is, rather, the general structure of his procedure. Frege's aim is to prove that the truths of arithmetic are analytic and so logical. To provide this proof, however, Frege revises formal logic in order to turn it into a possible ground for arithmetic. More specifically, he revises logic so that valid inferences can be formed concerning relations. This in turn enables inferences and logical laws to be formed concerning sequences. Such laws are then taken to govern the sequence of natural numbers, and to ground, among other things, the principle of mathematical induction. So, as Frege summarizes in his *Begriffsschrift*, the course he takes to connect arithmetic and logic is "first to seek to reduce the concept of ordering in a series [*Reihe*] to that of *logical* consequence, in order then to progress to the concept of number" (which will actually be defined by Frege in the *Foundations*) (FR 48 / BS x [Preface]).

Now general truths about sequences are "dependent on characteristics of two-place relations", such as "( ) is less than ( )" or "( ) is the immediate successor of ( )".[32] Such truths require, therefore, the introduction into logic of the argument-function distinction. This very distinction is, however, imported into logic from mathematics. So, in order to prove that arithmetic can be derived from *logic*, Frege first draws on *mathematics* in order to reform logic. In so doing, of course, he renders his whole procedure circular – or at least partly so, since he imports the argument-function distinction into logic in order to prove arithmetical truths that are not themselves taken for granted at the start of the proof.

By importing this distinction from mathematics into logic, Frege assumes that it can belong to logic without deriving it from the nature of *logic* itself (or

even from primitive logical truths, which are as far back as he can go). Yet one might think this an unfair criticism to make, since, as Weiner notes, Frege's general remarks about functions in texts such as *Begriffsschrift* and "Function and Concept" do not actually form part of his logical theory – they are not themselves laws of logic – but "are designed to be elucidatory remarks in a propaedeutic to the logic" itself.[33] The notion of a function, however, is a "basic" one underlying his logic, since all judgeable content that figures in logical laws and inferences must be clearly divided into a function and one or more arguments (and the symbols used must make this distinction clear, as in e.g. Φ (A)).[34] In the *Begriffsschrift* Frege states that it does not matter precisely *how* such content is divided – that is, which part is taken to be the function and which the argument – "so long as function and argument *are fully determined*" (FR 68 / BS 17 [§ 9], emphasis added). He does not show, however, that judgeable content must divide *itself* in this way by its very nature; he simply assumes that it can be so divided, and that it is useful for his broader purpose of proving arithmetic to be analytic to divide it in this way. In Hegel's logic, categories are not simply assumed to be dialectical, but prove themselves to be so. In Frege's logic, by contrast, judgeable content does not prove itself to contain the argument-function distinction. That distinction thus remains an assumption that Frege introduces into logic in order to fulfil his overall aim. Once again, therefore, despite his eagerness to presuppose as little as he can, Frege proves to be much more question-begging than Hegel.

CONCEPT AND OBJECT

The distinction between function and argument is closely connected to another distinction simply assumed by Frege, namely that between *concept* and *object*. Indeed, Beaney and Weiner both suggest that the latter distinction (first made by Frege in the *Foundations*) is based directly on the former.[35] In §9 of Frege's *Begriffsschrift* (1879), in which the distinction between a function and an argument is explained, functions are not explicitly associated with concepts, though the argument of a function is identified at one point with an "object" (*Gegenstand*) (FR 66 / BS 15). Concepts are said to be functions, however, in the essay, "Function and Concept" (1891). More specifically, a concept is said to be "a function whose value is always a truth-value" (FR 139 / FB 11).

The "value" of a function is the result yielded when the empty places in the function are filled with specific arguments. So, for example, when the function "$2(\ )^3 + (\ )$" is supplied with the argument "1", the value yielded is the number 3 (FR 134 / FB 6). When, however, the function of an *equation*, such as "$(\ )^2 = 1$" is supplied with an argument, the value yielded is what Frege calls a "truth-value". So, in this case, if the argument supplied is "1" or "-1", the truth-value yielded is "the True" (*das Wahre*), since the equation is true either way; but if

the argument supplied is any other number the value yielded is "the False" (*das Falsche*) (FR 137-9 / FB 9-11). A *concept* is thus a function that together with its argument forms a thought or judgeable content whose value is the True or the False; or in Frege's own words, "a concept is a function whose value is always a truth-value" (FR 139 / FB 11). Strictly speaking, a concept, for Frege, is a logical rather than mathematical function; but an equation containing the latter, he explains, can be converted into a judgement (or judgeable content) containing the former. So, for example, the equation "$x^2 = 1$" – that is, "$x = \sqrt{1}$" – is true when x is -1, and so, too, is the judgement "x falls under the concept: square root of 1". The concept "[is a] square root of 1" is thus simply a non-mathematical form of the function "$( ) = \sqrt{1}$".

In the *Foundations* (1884), in which Frege first gives his definition of the word "number", concepts are not conceived explicitly as functions whose values are truth-values.[36] Nonetheless, concepts are clearly modelled on functions and are understood in a way that is compatible with that found in "Function and Concept". Frege insists that "the distinction between concept and object must be kept in mind", and he states that "a simple concept", just like a relational concept, "always demands completion [*Ergänzung*] to become a judgeable content" (GA 23, GA 103 [Intro., § 70]). A concept in the *Foundations*, like a function in the later essay, is thus "incomplete, in need of supplementation, or unsaturated" (FR 133 / FB 5); and when it is supplemented by an argument, it forms a content that can be true or false.

If a concept in the *Foundations* is incomplete by itself, or what Frege later calls "predicative" (FR 182 / ÜBG 48), an object (from a logical point of view) is "independent" or "self-standing" (*selbständig*) and so, as he later puts it, "fully complete in itself" (*völlig in sich abgeschlossen*).[37] An object, in other words, just is what it is and so is "equal to itself" (*sich selbst gleich*) (FR 143 / FB 16). This is why Frege will argue in the *Foundations* that no object falls under the concept "*not* identical with itself" (GA 107 [§ 74]). Of course, since all thought is governed by the principle of identity, even concepts, which are incomplete, must have an *identity* of their own; indeed, their distinctive identity is what makes them logically distinct from objects. Objects, however, are fully and explicitly self-identical, in a way that concepts are not: their identity as objects consists, as it were, precisely in being self-identical. This in turn explains why, in Frege's view, we must be able to recognize objects as the *same again* (*wiedererkennen*), and perhaps why only objects, not concepts, can be regarded as equal to and the same as one another (GA 94 [§ 62], and FR 175 / ASB 28). (Concepts can be equal to one another only insofar as the same objects fall under each one [FR 175-7 / ASB 29-31].)[38]

Objects, for Frege, include numbers, as we have noted (see 2: 40), and, bizarrely, in "Function and Concept" truth-values are also declared to be objects (FR 140 / FB 13). An "object" is thus not merely something in space and time;

it is anything that is logically complete in itself (including things in nature conceived *as* logical "objects"). Furthermore, it is something utterly singular and one-of-a-kind: in Resnik's words, it is that which "can be referred to by a *singular* term and can be referred to only by a singular term".[39] An object-word, as opposed to a concept-word, is thus always preceded by "the singular definite article": *the* (if it is preceded by an article at all) (FR 184 / ÜBG 50).[40] This article indicates that each object is unique and is the only one that is *the* planet Neptune, *the* first emperor of Rome or *the* number five. As Frege puts it in the *Foundations*, therefore, "an object does not occur repeatedly": it does not occur anywhere other than where it is, and does not take the form of different instances of itself, but is just *the* singular object it is (GA 85 [§ 51]).[41] The definite article indicates that what is at issue is "a definite, individual object" (GA 71 [§ 38]).

By contrast, a concept-word is preceded by the indefinite article, *a* – or lacks an article altogether – and so does not signify any specific, individual thing; the following are thus concepts: "being a planet" (or "( ) is a planet"), "being an emperor" and "being a number" (see e.g. GA 85 [§ 51]). A concept-expression, for Frege, can itself contain an object-word, as in "being *a* moon of *the* planet Jupiter"; nonetheless, the difference between concepts and objects, and between their respective expressions, remains definitive. If, therefore, we say that "the morning star is *the* planet Venus", we are not subsuming an object under a concept, but we are identifying one object with another; that is, we are saying that the objects denoted by the two object-expressions are in fact the same thing. On the other hand, if we say that "the morning star is *a* planet", we are subsuming an object under a concept. (As Frege explains, in his essay "On Concept and Object" [1892], we can convert the statement of identity between "the morning star" and "Venus" into one that subsumes "the morning star" under a concept by recasting the statement as follows: "the morning star is *no other than* Venus". In this case, "( ) is no other than Venus", or "( ) is *an* object that is no other than Venus", is a concept- rather than an object-expression, albeit one under which only one object falls [FR 183 / ÜBG 49].)

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An important consequence of Frege's sharp logical distinction between object and concept is that every judgement connecting the two is a singular judgement. This is because it subsumes a *singular object* – there is no other kind – under a concept: it states that *the* (or *this*) such and such has a certain property. As Frege writes in the *Foundations*, "a concept is for me a possible predicate of a singular judgeable content, an object a possible subject of such a content" (GA 98 [§ 66n]). This sheds retrospective light on Frege's introduction of quantifier notation in his *Begriffsschrift*.[42]

Perhaps the most familiar syllogism in logic proceeds as follows: "All men are mortal; Socrates is a man; therefore Socrates is mortal". Expressed more abstractly, using variables, this syllogism has the following form: "all *a* are *b*; *c* is an *a*; therefore *c* is *b*". In his reformed logic, however, Frege does away with the plural subject, "all *a*", in the major premise. He does so by understanding the statement "*all* a *are* b" to say that "a *is* b, *whatever* a we take" (see FR 69 / BS 19 [§ 11]). More precisely, he understands it to say that "whatever *x* we take, if this x is a, it is also b"; or, to put it another way, "for *any* x, if x is a, then x is b". The expression "for any x" is the quantifier expression in this reformulated statement, and Frege assigns it a specific symbol in his new concept-script.[43] My interest here, however, is not in Frege's symbolic notation, but in the fact that he converts what is usually understood to be a *universal* judgement about *all a* (namely that all a are b) into a *singular* judgement about any *individual x* that is an *a* (namely that this x is b). Commentators on Frege claim that there are logical advantages to be gained from his introduction of the universal quantifier, "for any x".[44] To my mind, however, what is most significant about his supposed logical breakthrough is not the quantifier itself, but the fact that it allows him to make every judgement about objects logically *singular*: for this clearly conforms to his view that such a judgement, or "judgeable content", is ultimately about an object that is itself irreducibly singular.[45]

Of course, I cannot claim that Frege introduces his quantifier in the *Begriffsschrift* primarily *because* he regards objects, and statements about them, as singular. After all, he first draws his sharp distinction between object and concept in the *Foundations*, which was published in 1884, five years after the *Begriffsschrift*. Commentators agree, however, that this distinction is indebted to the one he draws in the *Begriffsschrift* between an argument and a function; and in this text he explicitly identifies the argument of a function with an "object" (*Gegenstand*) (FR 66 / BS 15 [§ 9]). So there is some reason to think that he might have been moved to introduce the quantifier by the idea that objects are singular entities.

Be that as it may, even if Frege thought of quantifier notation before he thought of objects as singular, the two thoughts are inseparable in his logic: for by eliminating the universal subject, "all a", the quantifier necessarily turns every judgement about objects into a singular judgement about a singular object. Frege's use of the quantifier is thus by no means ontologically innocent: for it goes hand-in-hand with a conception of objects as unique individuals. This conception in turn is indebted to the function-argument distinction that Frege simply *assumes* he may import from mathematics into logic. This does not itself make Frege's conception of an object (or of a concept) erroneous, but it does mean that it lacks what Hegel would consider proper justification. Indeed, Frege himself admits that it cannot be justified: "decomposition into a saturated and an unsaturated part", he writes, "must be considered a logically primitive

phenomenon which must simply be accepted [*anerkannt*] and cannot be reduced to something simpler".[46] Of course, the idea that there *is* "something simpler", namely pure being, and that all logical distinctions must ultimately be derived from that, never crosses Frege's mind.

The close connection between Frege's introduction of the universal quantifier and his logical/ontological distinction between objects and concepts also explains his otherwise puzzling claim that "the words 'all', 'any' [*jeder*], 'no', 'some', are prefixed to concept words" alone, not to object-words (FR 187 / ÜBG 53). Frege notes in the *Foundations* "that at first sight the proposition 'all whales are mammals' appears to be about animals, not concepts" (GA 82 [§ 47]). He regards this appearance as misleading, however, for the following reason. If we are actually to say something about animals, we have to speak about designated individual objects: about *this* one and *this* one. "In general", he writes, "it is impossible to speak of an object without in some way designating or naming it"; that is to say, we have to pick the object out and say something specifically about *it*. The problem, however, is that by itself "the word 'whale' does not name any individual creature [*Einzelwesen*]"; and the same is true, in Frege's view, of the expression "all whales". To say simply that "all whales are mammals" is thus not to designate any *individual* thing and to say something about it: it is not to say of any one thing "this is a whale and therefore a mammal". As Frege writes, if one asks which individual animal is being spoken of when we say "all whales", "one cannot point to a single one".

This point is crucial and merits repeating: for Frege, the phrase "all whales" does not point us toward any *individual* object; but it is only by pointing to an individual object that we can talk about an *object* at all, for every object, as such, is individual. Accordingly, Frege claims, even if we were looking directly at a whale, the simple judgement that all whales are mammals would tell us nothing about it. It would do so only if we were to add specifically that *this* object before us is a whale.

In Hegel's view, which is not derived and justified until the doctrine of the concept, "*all* are all *individuals*" (alle *sind alle* Einzelnen); one can thus designate and speak about all individuals together (SL 611 / LB 132). Accordingly, to say that all whales are mammals is to say something at one go about every individual whale. This is because all are *alike* in being individuals. In Frege's view, by contrast, one cannot designate and speak about all individuals at once, but one must do so *individually*, precisely because each is an irreducibly individual object. For the proposition that "all whales are mammals" to be about objects, therefore, it must be converted into a proposition about *one* singular thing at a time, namely the one we designate. This is achieved by means of the quantifier expression "for any x". The proposition, in Frege's reformulation (but using words rather than symbols), thus becomes: "for any x, if x" – that is, "the x we designate" – "is a whale, then x is a mammal". The

expression "for any x" indicates that the claim it precedes has *universal* scope; but the claim itself is in the form of a conditional judgement about one *singular* object (indicated by a variable), namely "*the* x we designate" or "*this* x". Similarly, as Frege notes in "On Concept and Object", the claim "all mammals have red blood" becomes the conditional claim "if something [*etwas*]" – that is, *this* something – "is a mammal, then it has red blood" (FR 186 / ÜBG 52-3); and the same must happen to the proposition "*a* mammal has red blood".

Note that in the expression "if x is a whale", the word "whale" is preceded by the indefinite article and so is a concept-word. The same is true of the word "mammal" in the expression "x is a mammal". The whole proposition, "if x is a whale, then x is a mammal", thus states that if the singular object, indicated by the variable "x", falls under the *concept* "being a whale", then it also falls under the *concept* "being a mammal". That is to say, the proposition *subordinates* the first concept to the second. The proposition thus actually does two things at once. First, it says something about the designated *object*: namely, that it falls under one concept, if it falls under another. Second, it says something about the two *concepts* concerned: namely, that one is subordinate to the other.

Let us now translate Frege's proposition back into its original form: "all whales are mammals". For Frege, we recall, this proposition does not designate any individual object and say something about *it*. Since the proposition says nothing about any object, however, all it can be about are the concepts it contains. The proposition "all whales are mammals" thus subordinates the concept "whale" under that of "mammal", and does no more than that (see GA 81-2 [§ 47]). Accordingly, as Frege remarks in "On Concept and Object", the word "all" in that proposition is prefixed to a concept-word, *not* (as one might think) to an object-word (FR 187 / ÜBG 53). The expression "all whales are such and such" appears to direct us to a range of objects; but, for Frege, this appearance is misleading, for in fact it directs us only to the concept "being a whale". We are directed towards objects only if the expression is reformulated to include a variable, *x*, that stands for a designated object, that is, if it states: "for any x, if *x* is a whale, it is such and such". (We can, of course, replace the word "whale" in the original expression with a variable and obtain the judgement-form "all *a* are such and such". In this case, however, the variable stands for a concept, not an object.)

Note, by the way, that there is nothing unusual in the idea that expressions of the form "all whales are . . . " tell us something about the concept "being a whale". What is unusual is Frege's insistence that such expressions tell us nothing about any object. One would normally think that such expressions tell us about those *objects* that are whales, that is, objects conceived *as* whales. In Frege's view, however, they are about the concept "being a whale" alone. This in turn reflects his conviction that any proposition about an object must be a singular judgement of the form "the x" – "or this x" – "is such and such". To

become a proposition about objects, therefore, the universal judgement "all a *are* b" must be converted, as we have seen, into a conditional singular judgement preceded by the universal quantifier. This in turn reflects the sharp distinction Frege draws between a concept and an object (which is irreducibly singular). This distinction, which, we are told in the *Foundations*, "must be kept in mind" (GA 23 [Intro.]), thus underlies, or at least goes hand-in-hand with, Frege's introduction of the quantifier and his association of quantifying terms, such as "all" and "some", with concepts rather than objects.

Frege's distinction, however, is not without its complications. First, although objects themselves are not concepts, "being an object" *is* a concept, as is indicated by the indefinite article. An object is thus something that falls under the concept "being an object": it is something that "*is an object*". The word "something" (*etwas*), however, is simply another word for "object".[47] An object is thus not just *something*, but an *object*, that falls under the concept "being an object". Yet shouldn't we then say that an object is an object-falling-under-the-concept-"being-an-object" that falls under the concept "being an object"? And does not this get us into an infinite regress? Furthermore, doesn't such a regress render Fregean objects systematically elusive: for whenever we think we are thinking of *an object*, are we not in fact just thinking of the *concept* "being an object" under which something must fall to be an object? Frege puts a stop to such a regress, however, by insisting that the definite article "the", or the demonstrative pronoun "this", does designate an object (GA 85 [§ 51]). We do not designate an *object*, therefore, when we speak simply of *an* object, but our words refer only to the concept of "being an object"; to speak of an object, we must speak of *the* or *this* such and such. Strictly speaking, indeed, there is no such thing as "*an* object" at all; there is only *the* car in my drive, *the* large tree in my garden, and so on, each of which falls under the concept "( ) is an object".

This would appear to keep "object" and "concept", as Frege conceives them, neatly distinct (in spite of the fact that "being an *object*" is a *concept*). This distinction proves to be less neat, however, when we consider Frege's use and conception of *variables*. In the essay "Function and Concept", Frege claims that the variable x in an expression such as "$2x^3 + x$" indicates a number (conceived as an object) "only indefinitely" (*nur unbestimmt*) (FR 133 / FB 5).[48] Yet how is it possible to indicate an object *indefinitely*, if only the *definite* article picks out an object? Surely, to indicate an object indefinitely is not to indicate an object at all, but to indicate a concept – a point Frege himself makes in the *Foundations*: "'indefinite object' is only another expression for 'concept'" (see GA 82 [§ 47]). A variable, such as x, must therefore be a concept-sign, rather than an object-sign. This is also suggested in the essay "On Concept and Object". There, as noted above, Frege states explicitly "that the words 'all', 'any', 'no', 'some', are prefixed to concept-words" (FR 187 / ÜBG 53); but this means that the variable in the quantifier "for any x" must stand for a concept,

not for a possible object. Accordingly, Frege's reformulation of the universal judgement "all a are b" must actually take the following, rather complicated, form: "for the *concept* 'being x', if something – *this* something – is an x, and if it is a, then it is b". It is true that when expressed in Frege's concept-script this judgement does not include the word "any", so one might think that this protects the variable in such an expression from having to stand for a concept. Yet Frege's symbol for the quantifier clearly stands for the thought "for *any* x"; indeed, his own verbal equivalent of that symbol is "*whatever* one may take as its argument" (was *man* auch *als ihr Argument ansehen möge*) (FR 69 / BS 19 [§ 11]).[49] So it would seem to be unavoidable that the x in the quantifier stands for a concept: the expression "for any x" means "for the *concept* 'being x'", rather than "for any *object* x" – or, rather, the latter is just a roundabout way of saying the former.[50]

This interpretation of the variable in the quantifier expression does not, however, change the meaning of the judgement in which it appears; nor, indeed, does it undermine Frege's logical distinction between concept and object, for the variable after the word "any" is now taken to indicate a concept *instead of* an object. Things are not quite so straightforward, however, when we turn our attention to the variable in the expression that follows the quantifier: for that variable is clearly meant to stand for an *object*. The expression that follows the quantifier, "for any x", is "if *x* is a, then *x* is b", and it is obvious that the variable *x* occupies the *object* place in both the antecedent and the consequent in the expression. To stand for an object, however, the variable must stand for *the* object we designate – whichever one it is – because every object is *the* singular thing it is, or *this* thing, and is not just *an* object in general. Yet the variable *x*, as a *variable*, indicates only indefinitely the object we designate. Accordingly, *x indefinitely* indicates a *definite* thing: it indicates *a* "this". This means in turn that it stands for both an object and a concept at the same time.

In the expression following the quantifier, the variable *x* stands for *the* specific object we designate, whichever that is. Yet this object is not actually specified and so is not something definite after all; this means in turn that it is not in fact an *object* (since every object is definite). Since what the variable stands for is not an object, however, it must be a *concept*. The variable x thus stands, not for *the* object we designate, but rather just for "being an object", "being an x". Yet this is not the whole truth, either: for in the expression following the quantifier, the variable x stands in the object places, and, as such, it clearly stands for *the* object we designate, whichever that is. As stated at the end of the last paragraph, therefore, the variable stands for both an object and a concept at the same time. Accordingly, the expression "if x is a, then x is b" means *both* the following simultaneously: "if *this* x is a, then *this* x is b" and "if 'being an x' is subordinate to 'being a', then 'being an x' is subordinate to 'being b'". The difference between "object" and "concept" is, of course, preserved in

these two different formulations. It is, undermined, or at least put under pressure, however, by the fact the expression of which they are formulations *means both at the same time*. This in turn is due to the irreducibly ambiguous character of the variable *x* that occupies the object places in it.

Note, by the way, that Frege's variable is here reminiscent of Hegel's "this" (*Dieses*) that is meant to "express something completely determinate", but that in fact expresses something wholly universal since "each and every something is just as much a 'this' as it is also an other" (see SL 91 / LS 112).[51] Frege's variable *x*, however, is not just *meant* to be determinate; as it figures in the phrase following the quantifier expression, what it indicates is both definite *and* indefinite at once.

THE *CONCEPT* . . .

Now just as we can speak of *an* object, and thus of the *concept* of "being an object", so, too, we can speak (as we have just done) of *the* concept of such and such, and in so doing we make an *object* of a concept. As we read in "On Concept and Object", therefore, "the three words 'the concept "horse"' designate an object, but on that very account they do not designate a concept". "The concept *horse* is not a concept", as Frege famously puts it, because it is a definite and complete logical object, whereas a concept is incomplete or "predicative" and needs to be supplemented by an object (FR 184-5 / ÜBG 50-1).[52] Frege blames language for appearing to undermine the distinction between concept and object: "by a kind of necessity of language, my expressions, taken literally, sometimes miss my thought: I mention an object, when what I intend is a concept" (FR 192 / ÜBG 59). Note, however, that at this point Frege by no means abandons the logical distinction between the two; indeed, his insistence that "the concept *horse* is not a concept" shows that he sticks to it resolutely, even to the point of paradox.

The distinction between concept and object does not, however, mean that they are completely separate from one another. Indeed, they are joined together precisely by the predicative nature of the concept: for the latter cannot stand alone but must be completed by an object, and when this occurs a thought (or judgeable content) is formed (see FR 193 / ÜBG 60). If an object (or object-expression) combines with a concept (or concept-expression) to form a true judgement, the object can be said to "fall under" the concept. For Frege, indeed, "the fundamental logical relation is that of an object's falling under a concept" (FR 173 / ASB 25). So hydrogen "falls under" the concept of being "lighter than carbon dioxide", if the expressions "hydrogen" and "is lighter than carbon dioxide" together yield a true judgement. An object also falls under a concept if we say of "the concept *horse*" that it is "easily attained" (and this statement is true). In this case, we ostensibly bring one concept – that of being "a *horse*" –

under another, second-level, concept – that of being "easily attained" – and say that the latter characterizes the former. Yet, as we noted above, by using the phrase "*the* concept horse" we convert the *concept* "horse" (or "being a horse") into an *object* and regard "being easily attained" as its property (FR 184-5 / ÜBG 50-2). We thus in fact bring an object, not a concept, under a concept. By contrast, when we say, for example, that a whale is a mammal, we do not subsume an object or a concept *under* a concept, but we subordinate a particular concept ("being a whale") *to* a more universal one ("being a mammal").[53]

Frege follows Kant in insisting that there can be no ambiguity as to whether an object falls under a concept or not. Kant claims in "The Ideal of Pure Reason" in the first *Critique* that every thing is subject to the "principle of thoroughgoing determination". This is the principle "according to which, among all possible predicates of things, insofar as they are compared with their opposites, one must apply to it" (CPR B 599-600). That is to say, one from each pair of opposed predicates must apply to each thing. So each thing must either be or not be a tree, and must either be or not be a colour, and so on; or, in Frege's terminology, each thing must either fall or not fall under the concept "tree" and the concept "colour". For Kant, this principle is a transcendental one and is the condition under which alone an object of cognition is really, as opposed to just logically, possible. It must be distinguished, therefore, from the purely logical law of contradiction. The latter states that of two contradictory predicates only one *can* belong to a thing, or, in Kant's words, "no predicate pertains to a thing that contradicts it"; so if something is a tree, it cannot not be a tree (CPR B 190). The transcendental principle goes further than this and states not just what something can or cannot be, but what it *must* be if it is to be a possible object of cognition; so not only can a thing not not be a tree, if it is one, but it must *be* one or the other: a tree or not-a-tree.[54]

Frege adheres to the same principle, though he regards it as simply logical – and for that reason objective – rather than transcendental in Kant's sense: it lays down what must be the case in "logical space" without qualification, not just the condition under which something can be an object of cognition.[55] In the *Foundations*, for example, Frege says that logic demands of a concept its "sharp delimitation", that is, that "for every object it be determined whether it falls under the concept or not" (GA 107-8 [§ 74]).[56] So not only can an object not be not-A, if it is A, and not be A, if it is not-A, but it must be *either* A *or* not-A; there is no third option (see FR 7). Frege is thus a perfect example of what Hegel would consider to be a dogmatic "metaphysician" who assumes that "of *two opposed assertions* [. . .] one must be *true*, and the other *false*" (EL 69 / 98 [§ 32]). As we have just seen, Kant also insists that any object of cognition must be either A or not-A; in contrast to Frege, however, he thinks that pure logic alone permits a third option, for it allows something (that is not an object of cognition) to be neither A nor not-A. Indeed, he exploits this logical

possibility in his critique of the mathematical antinomies: for Kant, the "world" (which is not a possible object of cognition) is neither finite nor infinite in itself, since it has no being "in itself" at all but is a mere Idea of reason (see CPR B 532, and 1: 337). Nonetheless, to Hegelian eyes, Kant remains a metaphysician of the understanding, like Frege, since he does not countenance the speculative *unity* of determinations "in their opposition" (EL 131 / 176 [§ 82]).

Note, by the way, that Hegel does not reject out of hand the everyday conception of truth and falsity: he acknowledges that "it may certainly be correct [*richtig*] that someone is ill, or has stolen something" (or it may be incorrect) (EL 250 / 323 [§ 172 A]; see also 60 / 86 [§ 24 A2]). In this respect, he can agree with Frege that an object must either fall or not fall under a concept. Hegel does not accept, however, that "either A or not-A" is an ultimate logical or ontological law, for ultimately things are more complicated than this. It may be true that for everyday consciousness things are for the most part one thing or the other (though even here, there may be uncertainties: who was responsible for that accident? was that a penalty or not?). Yet when we examine the fundamental categories that structure the world, we encounter dialectical ambiguity everywhere. Every something (*etwas*), for example, is both something and not just something (because it is also *other* than something else); equally, it has its own identity "in itself", but that identity is open to determination by others. Finite things come to an end and so are resolutely finite; yet they are also moments of the process that Hegel calls the "true infinite" (and which is manifest in self-consciousness); and the movement in which the many ones repel one another is also the movement in which they attract one another.[57] Frege wants objects, concepts and the relations between them, indeed all logical language, to be "unambiguous" (*eindeutig*) (BS 108), but Hegel shows in the *Phenomenology* and the *Logic* that there is dialectical ambiguity throughout our experience and being itself. It is not unambiguously clear, therefore, whether the French Revolution falls under the concept "being an act of freedom" or "being an act of unfreedom and death", since, in Hegel's view, it turns into the second by being the first.[58] Similarly, it is not unambiguously clear whether a thing's qualitative limit distinguishes it from its other or makes it the same as its other, since, as Hegel understands it, the limit does the one in doing the other (see SL 98-100 / LS 122-4).

As I have argued in this study, Hegel does not simply presuppose such dialectic in the way that Frege just presupposes that objects either fall or do not fall under a given concept. He shows that dialectic is inherent in being (and thought) itself. Equally, Hegel does not simply presuppose that there is *no* sharp distinction between object and concept in the way that Frege assumes there *is*. Indeed, at the start Hegel does not presuppose objects or concepts at all in the senses in which they are conceived by Frege, Kant or speculative logic itself later in its course. He starts with the bare thought of indeterminate being and then derives more determinate categories from that indeterminate beginning –

categories that include those of "something", "quantum", "number" and, eventually, "concept" and "object" (*Objekt*). Since, at the start of logic, there is no distinction between indeterminate being and indeterminate thought, the categories that emerge in speculative logic belong to both being and thought. They are the categories in which we must think, but also the determinations of being itself. Indeed, they are the categories, or forms of being, through which alone there can be determinate finite things, quanta, numbers and *objects*. For Hegel, therefore, objects do not in truth "fall under" concepts that are logically distinct from them, but they are structured and constituted *by* categories.

The distinctive logical structure of a "concept" as such (rather than a category in general) and of an "object" (rather than just "something") becomes clear only when we reach the doctrine of the concept. It is only there, too, that we learn how mechanical, chemical and organic objects manifest the structure of "the concept", as well as that of the "judgement" and "syllogism" (all of which are ontological as well as logical forms). Yet even from the account of quality and quantity that we have examined so far, it is clear that ways of *being*, such as being something and being finite – and so being an "object" in a rudimentary form – coincide with logical forms or *categories*. To *be* something and to *be* finite is thus to exhibit the category or logical form of "something" and "finitude"; indeed, the category of something just *is* the form of being something. In this sense, Hegel's logic confirms Kant's insight that categories provide the structure of objects, and Hegel's speculative logic can be seen as a further development of Kant's transcendental logic (see SL 40 / LS 48). The difference between the two logics is that Hegel takes the categories to be inherent in *being itself* and to form it immanently *into* objects, whereas Kant thinks categories make possible only the objects of finite cognition, that is, objects of experience.

Hegel thus cannot be said simply to fuse together what Frege holds apart, because he does not start with determinate "concepts" (or categories) and "objects" at all. Nonetheless, as his logic unfolds, pure being takes the form of being determinate, being something and being finite, and in that sense makes rudimentary *objects* necessary. These forms of being coincide with the various *categories* inherent in thought. Indeed, the two are one and the same: the categories that belong to thought and being. What emerges in speculative logic, therefore, are categories that are immanently constitutive of objects, not concepts "under" which objects "fall". Thus, from a Hegelian point of view, not only is Frege's sharp distinction between objects and concepts presupposed without warrant (on the basis of the distinction between argument and function that is itself simply assumed to be appropriate for logic), but that distinction obscures the true relation, indeed identity, between objects and the categories that give them form.[59]

It is important to stress that, for Frege, concepts are not subjective mental representations, but are objective (albeit incomplete). They stand in objective

relations to one another that we do not control but must recognize and grasp, and they coincide (in one sense) with the properties of objects (both natural and logical).[60] Nonetheless, concepts *as such* are logically distinct from objects: they are not immanently constitutive of objects (like Hegel's categories), but objects "fall under" them. This latter idea, however, creates a significant problem for Frege that we have not so far considered.

THE EXTENSION OF A CONCEPT

In the *Foundations* (1884) the set of objects falling under a concept is said to form the "extension" (*Umfang*) of that concept: as Weiner puts it, a concept's extension "has as its members precisely those things of which the concept holds".[61] In the essay "Function and Concept" (1891), however, Frege equates the extension of a concept with the latter's "value-range" or "course of values" (*Wertverlauf*) (FR 139 / FB 12). In that text he also states that the *value* of a concept is a truth-value; so the value of the concept "is a horse" is "the True" for Dobbin, "the False" for Rex, and so on (FR 139 / FB 11).[62] If, therefore, we assume that a "value-range" is simply the range of values yielded by a concept for a range of objects, then it will comprise the specific set of *truth-values* distinguishing that concept: the True, the False, the True, and so on. The extension of the concept will thus itself be that set of truth-values. Yet this is hard to reconcile with the idea that a concept's extension encompasses the *objects* that fall under it: the "True" and the "False" are, indeed, "objects", for Frege, but they are surely different from ordinary objects (as we might call them), such as Dobbin, whose properties coincide with the concepts under which they fall, such as "being a horse". Frege's two conceptions of "extension" can be brought together, however, if we conceive of the value-range – and thus the post-1891 extension – of a concept as the set, not just of truth-values, but of *pairings* of objects and truth-values, that distinguish that concept from others. So, as Currie puts it, "the course of values (extension) of the concept *being German* would be the set of pairs (Frege, the true), (Napoleon, the false), (Plato, the false), etc.".[63] The extension of a concept, for Frege, thus always encompasses the objects falling under that concept; after 1891, however, those objects are paired with truth-values, whereas in 1884 they are not.

The idea of a concept's "extension" is central to Frege's conception of number in both the *Foundations* and the *Basic Laws*. Yet it also leads to Russell's famous paradox. To see precisely how this paradox arises, consider the following: namely that the extension of a concept does not always fall *under* the concept *of* which it is the extension. If, for example, we take the concept "( ) is a horse", then objects that fall under this concept include Dobbin, Arkle, Seabiscuit and so on; these objects are thus all horses and so belong to the extension of the concept. That *extension*, however, does not "fall under" the

concept "( ) is a horse", since it is patently *not* a horse itself. It is the set of objects – namely, horses – falling under the concept (or of pairings of such objects and truth-values), but it is not itself one of those objects. If we now take the concept "( ) is an extension", the case is different. The objects falling under this concept include all the extensions there are; these extensions taken together thus form the extension *of* the concept itself (with the addition of truth-values after 1891 and without them before then). This very extension, however, also falls *under* the concept concerned, since it, too, *is an extension*, just like the extensions it encompasses.

If we now take a third concept, "( ) is the extension of a concept under which it does not fall", it is clear that the extensions of concepts, such as "( ) is a horse", fall under this concept, since they do not fall under their respective concepts. These extensions together thus belong to the extension of our third concept. Yet what should we say of this latter extension? Does it fall under that third concept or not? Let us first assume that it does. In this case, it is the extension *of* our third concept and it falls *under* that concept, so it is the extension of the very concept under which it falls. Yet if it falls under our third concept, we have a problem, for it must then have the property designated by the concept itself; that is, it must be the extension of a concept under which it does *not* fall. So if it falls under the concept of which it is the extension, then the very concept under which it falls means that it does *not* fall under the concept of which it is the extension. We thus encounter the one thing Frege wishes to avoid at all costs: contradiction.

If we now assume that the extension of our third concept does not fall under that concept, then, of course, it is one of the many extensions that do not fall under the concepts of which they are the extensions. This in turn means that, like those other extensions, it falls under the concept "( ) is the extension of a concept under which it does *not* fall". That very concept, however, is the concept of which it is the extension. The extension thus falls, after all, *under* the concept *of* which it is the extension. So precisely by not falling under its concept the extension of that concept falls under it. Once again, we face contradiction – the paradox that Russell brought to Frege's attention in 1902.[64]

This paradox is generated by considering extensions, which encompass the objects falling under a concept, to be objects themselves and thus able to fall under concepts, including their own. This idea that an extension is itself an *object* is first introduced in the essay "Function and Concept" (FR 141 / FB 13).[65] Yet it is surely implicit in the *Foundations*, too, for in that work there are only two ontological possibilities: something is either a concept or an object.[66] Since the extension of a concept is not the concept itself, it must be an object; and since objects fall under concepts, extensions must also be able to fall under them. Russell's paradox thus in fact has its roots in the very idea of an *extension* introduced in the *Foundations*.[67] That idea, however, is itself made necessary by

the distinction between concepts and objects: for if objects are distinct from and fall under concepts, it is natural to think of the objects under one concept as forming a single set or "extension" together. Russell's paradox thus has its ultimate source in the distinction between a concept and an object (and so between a function and an argument) that Frege takes to be fundamental to his enterprise: it is the contradiction implicit in Frege's thought *from the start*.[68] To put it another way, Russell, the severe critic of Hegel, renders explicit the dialectic to which Fregean "understanding", despite its best intentions, inevitably gives rise. This dialectic, from a Hegelian point of view, should lead to the sharp distinction between concept and object being undermined and to the two being thought together as one. Objects would thus be constituted *by* concepts and categories (understood as ways of being) and would not "fall under" the latter. The idea of a concept's "extension" would thus disappear, and the paradox it generates would thereby be resolved. Frege's own response to the paradox, however, was not to undermine the distinction between concept and object or to do away with the associated idea of an "extension". His solution, as Beaney puts it, "was simply to outlaw the applicability of concepts to their own extensions", so that the question whether extensions do or do not fall under the concepts of which they are the extensions would never arise (FR 8). Whether this prevents the emergence of further contradictions, I shall leave to others to judge. It is, however, inconsistent on Frege's own terms: for if extensions are objects, and if "the fundamental logical relation is that of an object's falling under a concept" (FR 173 / ASB 25), how can one deny that extensions are capable of falling under their own concepts?

Russell's paradox is generated by Frege's idea of a concept's extension. After being shown the paradox in 1902, however, Frege does not immediately abandon the idea, but he simply denies that an extension can fall under the particular concept to which it belongs. The idea of an extension thus remains integral to his logic and to his derivation of arithmetical truths from logic in the second volume of the *Basic Laws* (1903) (see FR 288). As noted above, the idea also plays a crucial role in Frege's account of number in the *Foundations*, published eighteen years before Frege received Russell's fateful letter. One should remember, however, that in that text an extension is simply the set of objects falling under a concept and does not include truth-values. More importantly, one should remember that the idea of an "extension" rests on the sharp distinction between objects and concepts that Frege simply takes for granted. The problematic idea of an "extension" is thus itself a mere presupposition, as Frege himself concedes: in Weiner's words, it is, for Frege, an "assumption [. . .] fundamental to logic" and "must simply be taken as a primitive logical notion".[69] Indeed, Frege takes it to be so self-evident in the *Foundations* that he does not even bother to define it, but instead says simply: "I assume that it is known what the extension of a concept is" (GA 101 [§ 68n];

see also 137 [§ 107]). We see once again, therefore, that Frege's account of number – whatever other merits it may have – rests, by his own admission, on assumptions that are not, and cannot be, justified. Frege's account of number is thus question-begging in a way that Hegel's is not.

FREGE'S CONTEXT PRINCIPLE

Before we move to look directly at Frege's account of number, there is one more Fregean assumption to mention. This is his well-known "context principle". Commentators disagree on whether Frege abandons this principle in his later work;[70] but it clearly plays a major role in the *Foundations* in preparing the way for the definition of "number". The principle states that "the meaning of a word must be asked for in the context of a proposition [*Satz*], not in isolation" (GA 23 [Intro.]), and so, Dummett argues, its introduction marks the point at which the "linguistic turn" that inaugurates analytic philosophy first occurs.[71] The principle also accords with Frege's insistence on the primacy of *judgements* (and, indeed, of inference) over concepts. As Brandom notes, "at least until 1891, Frege clearly regarded the claim that concepts can only be understood as the products of analysis of judgments as one of his most central insights"; and the context principle makes a similar claim about concept- and object-*words* and *sentences*.[72]

Hegel criticizes pre-Kantian metaphysics for taking judgement for granted and not investigating "whether the form of the judgment could be the form of truth" (EL 66 / 94 [§ 28 R]). Such metaphysics does not ask, therefore, whether judgements, such as "the world is infinite" or "the soul is simple", can disclose the true character of things. Kant argues that theoretical judgements must relate to objects of a priori or empirical intuition to be capable of truth or falsity, but he, too, just assumes that thought is minimally judgement and that concepts are "predicates of possible judgments" (CPR B 94).[73] Hegel, by contrast, makes no such uncritical assumption but *derives* the form of judgement in the course of his logic from the "concept" and ultimately from indeterminate thought and being. He thus demonstrates, rather than takes for granted, that thought (and being) must take the form of judgement at a certain point (see SL 550 ff. / LB 58ff.). In the process, however, he also shows that "the proposition, in the *form of a judgment*, is not suited to express speculative truths", because it cannot express the dialectical transition or "movement" of one category into its opposite (SL 67 / LS 82). He is forced, therefore, to develop what he calls "speculative propositions" in order to articulate that dialectic in its dynamism.[74]

Hegel would thus criticize Frege for presupposing the primacy of judgement, just as he criticizes Kant for presupposing "the *various kinds of judgment*" and for basing the categories on them (EL 84 / 117 [§ 42 R]). He would also criticize Frege for dogmatically insisting on the primacy of sentences over the words

(and meanings) that they contain. Hegel does not deny at all that thought is inextricably connected to language. Before the start of his philosophy, in the second preface to the *Logic*, he maintains that the "forms of thought are first set out and stored in human *language*" and that "many of the prepositions and articles already pertain to relations based on thought" (SL 12 / LS 9-10). Then in his *Encyclopaedia Philosophy of Spirit* he demonstrates that "it is in names that we *think*" (EPM 199 / 278 [§ 462 R]). For Hegel, therefore, we come to understand categories as we learn to speak, and so "language enables the human being to apprehend things as universal" (EPM 57 / 80 [§ 396 A]). Accordingly, Hegel's conception of thought and language is clearly compatible with the idea that we can, and do, learn the meanings of concept-words (and object-words) in the context of sentences.

Yet Hegel would not accept Frege's dogmatic assertion that one *must* always determine the meaning of words "in the context of a proposition". In particular he would not accept that this principle governs philosophy. The reason why not should be familiar to us: philosophy may not presuppose anything determinate about thought at the outset, but must discover what thought – starting from its indeterminate being – determines itself to be. We may not allow the structure of sentences to determine how categories are to be conceived, any more than we can allow presuppositions about the "laws of thinking" to do so (SL 23 / LS 25).

This is not to deny after all that we think in language; but presuppositionless thought must hold at bay the familiar meaning of words, such as "becoming", "something", "infinite", and allow the logical structure of the corresponding categories to emerge immanently from pure being. As Hegel writes in the *Logic*, when we consider how a category is to be understood, "this cannot be an issue of confirmation [*Bestätigung*] based on the *authority* of ordinary understanding; in the science of the concept, its content and determination can be proven solely by the *immanent deduction* which contains its genesis" (SL 514 / LB 12).[75] We must also keep presuppositionless thought free of the prior influence of the *proposition* (*Satz*): for, like the judgement, it would impose on thought a fixed distinction between subject and predicate that we may not assume at the start (and with which dialectic actually proves to be at odds). Frege analyses a proposition or judgement into an argument and function, rather than a logical subject and predicate; from a Hegelian point of view, however, that distinction may no more be presupposed than that between subject and predicate. In a truly self-critical logic one may not allow the development of thought, and the logical structure of the categories that emerge, to be governed by the structure of the proposition at all; and so one may not seek the meaning of logical terms – category-words – by examining them in the "context" of a proposition. One must derive such meaning – the categories themselves – from the sheer indeterminacy of being and thought.

It is noteworthy that in one sense Frege himself wishes to free logic from the constraints of language: "it is a task of philosophy", he suggests in his *Begriffsschrift*, "to break the power of words over the human mind [. . .] by freeing thought from the taint of linguistic means of expression" (FR 50-1 / BS xii-xiii [Preface]).[76] Hence his concern to devise a "concept-script" that dispenses with the expression of anything "that is without significance for *logical inference*" (FR 49 / BS x [Preface]). Yet Frege does not think, as Hegel does, that logic should begin by breaking the power of the proposition and of judgement – and, indeed, of formal inference – over thought (as it should break the power of anything that would determine the character of thought in advance). For Frege, in the *Foundations* at least, the idea that thought occurs in propositions and judgements, and that the meaning of a word such as "number", must be sought in the "context of a proposition", is a founding principle that is taken for granted dogmatically at the start.

To recapitulate: Frege's account of number in the *Foundations* rests on the following assumptions, all of which for Hegel, and some of which for Frege himself, lack ultimate justification. These are: a) that logic, as Frege conceives it, lays down rules that govern all thought; b) that arithmetic also governs "everything conceivable", since everything conceivable is countable; c) that the distinction between argument and function is central to logic; d) that the related sharp distinction between object and concept is central to logic; and e) that the context principle holds and thus that the meaning of the word "number" must be determined by examining it in the context of appropriate propositions. Since Frege's account of number presupposes all the above, it is clearly more question-begging, and so less justified, than Hegel's radically presuppositionless account, even if it is deemed by many to be highly fruitful.

In the next chapter we will look closely Frege's account of number itself. We will consider first his critique of the traditional (Greek) conception of number, which coincides in part with Hegel's conception, and then how he arrives at his own logical definition of number. In the course of the chapter we will explain in detail why, from a Hegelian point of view, Frege's conception of number is problematic and so cannot be taken to supersede Hegel's conception.

CHAPTER FIVE

Excursus: Hegel and Frege on Number

FREGE'S CRITIQUE OF THE TRADITIONAL (GREEK) CONCEPTION OF NUMBER

Frege remained unaware of Hegel's derivation of number from quantity and, ultimately, from pure being. In the *Foundations*, however, he provides a detailed critique of a conception of number, close to Hegel's, that has been popular since the Greeks and, according to Dummett, is shared by Husserl and Cantor among others. Indeed, Dummett contends, Frege refutes the theory underlying this conception "brilliantly, decisively and definitively".[1] From a Hegelian point of view, however, Frege's critique rests, like other aspects of his thought, on specific assumptions that are never properly justified.

According to the popular conception, Frege tells us, a number is "an aggregate" or "plurality", and more specifically "an aggregate of units [*Einheiten*]" (GA 60-1 [§ 28]). His first objection to this conception is that it excludes 0 and 1 from the concept of number, for neither can be said to contain a *plurality*. In his view, 0 and 1 are numbers in the same sense as 2 and 3, so what does not apply to them "cannot be essential for the concept of number" (GA 79 [§ 44]); a number thus cannot be in its essence a plurality. Yet in making this criticism Frege takes for granted at the start that 0 and 1 are, indisputably, numbers, even though he would have known that this has not always been beyond dispute: for Aristotle, for example, the smallest number in the strict sense is 2.[2] From Hegel's point of view, however, the more significant problem with Frege's criticism is that it makes assumptions at all about number before providing a definition of the latter.

For Hegel, as we know, the task of philosophy in modernity is to think *freely* without being guided by prior assumptions about what is to be discovered. We

cannot, therefore, allow ordinary experience, or sciences other than philosophy, to determine in advance what is to count as a "number", a "state" or, indeed, "being" itself: for to do so would be to limit, dogmatically, the freedom of thought. Speculative philosophy can thus do no more than think pure being and wait to discover what, if anything, is made necessary by it. In the process, as we have seen, we learn that quantity and then number are made necessary by pure being and that number is a determinate quantum containing a determinate *plurality* of units. To be more accurate, we discover that being makes necessary a determinate plurality of units and we then *name* that plurality "number". That name is drawn from ordinary experience because the ordinary, non-philosophical notion of "number" best matches what philosophy has discovered. Philosophy, for Hegel, can thus draw the *names* for its categories from other areas of experience, provided that those *categories* themselves are derived immanently and without external – and so unjustified – interference. As Hegel puts it in the *Logic*, "philosophy has the right [*Recht*] to choose such expressions from the language of ordinary life [. . .] as *seem to approximate* the determinations of the concept" (SL 628 / LB 154);[3] but common life, or non-philosophical science, does not have the right to dictate how philosophy, which is meant to be thought in its most unconstrained form, must conceive of the categories it associates with those expressions. Philosophy, for Hegel, is thus not required to stick to what "number" or "the state" are *assumed* to be in ordinary life or science, but it attaches new, non-question-begging conceptions to those names. In so doing it revises and "purifies" the concepts and categories of everyday life and discloses what number and the state are *in truth* (SL 17 / LS 17).

In Hegel's view, therefore, the fact that 0 and 1 are held by many to be numbers just like 2 and 3 cannot be permitted to constrain what philosophy understands a number to be. It is, rather, the task of philosophy to discover how the concept of number is properly to be understood and *whether* 0, 1, 2 or 3 are truly numbers. Hegel's logic shows number to be a determinate *plurality* of units (as many others, such as Euclid, have also claimed).[4] Moreover, a "plurality" is produced when the one (*Eins*) continues beyond itself in *another* one (SL 135-6 / LS 171-2). Strictly speaking, therefore, neither 0 nor 1 is a number proper, but the smallest number, as Aristotle maintains, is 2 (or, indeed, 3, if "many" is understood to be more than just one and another one).

Frege's rejection of the philosophical idea that a number is a plurality, on the grounds that this would prevent 0 and 1 from being numbers, is thus illegitimate to Hegelian eyes, because it assumes from the start that 0 and 1 are numbers "in the same sense as 2 and 3" (GA 79 [§ 44]). Frege thereby allows an assumption from outside philosophy to determine in advance how philosophy, which is meant to be free, may or may not understand the nature of number. Yet in defence of Frege one can, of course, point out that he does not share Hegel's conception of philosophy. He is not seeking, as Hegel is, to discover *without*

prior assumptions what is inherent in thought and being and what number may prove to be. His concern, as he makes clear in the *Foundations*, is, rather, "to grasp the concept of number in such a way that it is *useful for science*", that is, for arithmetic (GA 90 [§ 57], emphasis added). It makes good sense, therefore, for him to seek a conception of number tailored to the numbers with which arithmetic operates and so to presuppose that 0, 1, 2 and 3 are all numbers in the same sense. This is true enough; but then Frege cannot claim that the resulting conception of number has more *justification* than any other: for the very assumption he makes about 0 and 1 renders that conception question-begging, and so deprives it of the ultimate justification that, for Hegel, comes only from being derived from an indeterminate, presuppositionless beginning. That assumption also leaves Frege's initial criticism of the popular conception of number without justification.

Note that, from a Hegelian viewpoint, Frege's criticism is also mistaken: for the idea that a number is a plurality does not prevent 0 and 1 from being considered as numbers, even if neither is a number in the full sense. Numbers, for Hegel, are unities containing a plurality of mutually external units that can be subtracted from the numbers to which they belong. 0 and 1 can therefore both be generated by subtraction from a natural number proper, such as 3, 4 or 5. Hegel's conception of number thus does nothing to stop mathematicians working with 0 and 1, and the same must be true of the similar popular conception. Hegel's conception does mean, however, that, logically, 0 and 1 are not the origin of the other numbers, but are derivative from them. The "one" (*Eins*) that gives rise to quantity, and thus eventually to number, in Hegel's logic is thus not itself the *number* 1, but rather the qualitative one.[5]

Frege's second objection to the popular conception of number is directed against the idea that the units encompassed by a number could be "*equal to one another*" (*einander gleich*) (GA 66 [§ 34]). His objection rests on various arguments, the first of which proceeds as follows. If the units in a number are equal (or, as Hume puts it, "entirely similar"),[6] then any objects that are to be counted must themselves be treated as equal: each must be seen as simply *one* of things to be counted. This in turn means that we must disregard, and abstract from, the properties that make the things distinct: for "the properties, through which the things distinguish themselves, are a matter of indifference to and alien to their number" (GA 67 [§ 34]). In Frege's view, however, if we abstract from the distinctive properties of things, we are not left with a set of identical featureless units; rather, what we get is a general *concept* under which the things fall. If I am faced with a white cat and a black cat, and I disregard their different colours, I am left with the concept "cat", and if I abstract from all properties of the cats I am left with the bare concept "object" or "thing".

Abstracting from the distinctive character of things does not, therefore, turn them into countable units: it does not, as Dummett puts it, "create abstract

objects".[7] It just switches our attention from the things to the property or concept they have in common. The reason why this is, is clear. For Frege, objects by their nature are *distinct* from one another: they are unique singular things. Accordingly, "two objects are *never* completely equal" (GA 66 [§ 34]). They cannot, therefore, be reduced to equal, countable units and still remain objects. So if we disregard the particular character of objects, we actually lose sight of those *objects* altogether and are left with a *concept* instead. A number of objects thus cannot comprise a set of equal units, as the popular conception of number would have it. Frege's argument against that conception rests, however, on his assumption that objects are irreducibly singular in contrast to general concepts. Since this assumption is never properly justified by Frege, his criticism of the popular conception of number is not justified either.

In § 35 of the *Foundations* Frege repeats the claim that different things cannot be made equal; he goes on to say, however, that even if they *could* be made equal, "one would no longer have things [*Dinge*], but only one thing [*nur ein Ding*]" (GA 68). Each thing is different in some way from every other thing; if, therefore, two things were *equal* – in being bare units – they would not actually be two things, but they would "melt irretrievably into one" (GA 75 [§ 41]).[8] In putting forward this argument, of course, Frege tacitly presupposes Leibniz's well known principle of the "identity of indiscernibles". In his *New Essays on Human Understanding* (completed by 1704-5, but published posthumously in 1765), Leibniz's spokesman, Theophilus, argues that "if two individuals were perfectly similar and equal and, in short, *indistinguishable* in themselves, there would be no principle of individuation. I would even venture to say that in such a case there would be no individual distinctness, no separate individuals".[9] In other words, there would not be two at all, but only one. For Leibniz himself this principle applies to individuals "in nature" (that is, substances), but not to the components of a number which are, indeed, equal units.[10] This is what enables him to say that there cannot be "two individuals who are exactly similar, or who *differ only numerically*".[11] Frege, however, also directs Leibniz's principle against the idea that a *number* contains many equal units. If a number were to contain equal units, he claims, it could not contain many of them, because they would in fact coalesce into a *single* unit.[12]

In § 36 Frege then quickly moves on to the corollary of this argument: if there are, indeed, *many* units, and not just one, then those many must be different and so cannot simply be identical *units*. Frege makes this point while considering a passage from W.S. Jevons' *The Principles of Science*. In that passage Jevons conceives of the number 5 as a set of units or ones, that is, as "1 + 1 + 1 + 1 + 1"; but he adds that "each of these units is distinct from each other", so they might, "if requisite", be symbolized like this: "1´+ 1´´ + 1´´´ + 1´´´´ + 1´´´´´".[13] Frege affirms that the units should, indeed, be given different signs to mark their difference, but he notes that they could be

symbolized just as well, indeed better, like this: "a + b + c + d + e". In this case, however, all equality between them is lost and they cease being mere units or ones at all: "the one slips through our hands" (*so zerrinnt uns die Eins unter den Händen*), and "we get the objects with all their particularities" (GA 69 [§ 36]). So if we are to think of a number as a set of units, and those units are to be different and not just to combine into one, they cannot be simple units but must be distinct objects. The popular idea that a number contains a plurality of identical *units* is thus unsustainable.

To summarize: a number cannot contain many identical units, because if the units were identical, they would coalesce into one; and if there were many of them, they would have to be different and so would not just be identical units. That is to say, if they are ones they can't be many, and if they are many they can't be ones. Either way, in Frege's view, *there cannot be many ones* (as the tradition, from the Greeks to Husserl, would have it).[14]

In § 38 Frege makes a related but subtly different point: there cannot be many units or "ones", not just because they would coalesce, or not be ones in the first place, but because one is itself a number and therefore a *unique object*. As Frege writes, "one says 'the number one' and indicates with the definite article a definite individual object of scientific investigation". Accordingly, the word "one" (*Eins*), as the name of that individual object, cannot be given a plural form: "there are not various number ones, but only one", just as there is only one "Frederick the Great" (GA 71; see also 80 [§ 45]). The expression "1 + 1" does not, therefore, designate two objects, *each* of which is *the* number one – for there is only one number one – but it designates just one number, namely *the* number 2. The number designated by "1 + 1" is thus the *same* as that designated by "2", and the equation "1 + 1 = 2" is correct.[15]

In contrast to an object-word a concept-word is capable of being given a plural form: we can speak, for example, of "all whales" or "many whales", and when we do so we bring to mind a concept – "( ) is a whale" – and think of several objects falling under it. When we talk about "many ones" or "many units", therefore, we are in fact treating the words "one" and "unit" as *concept*- rather than object-words. The phrase "many ones" thus refers not to a set of identical objects called "ones" – a thought that Frege deems unsustainable – but to a set of distinct objects that fall under the concept "( ) is a one" or "( ) is a unit".

There is, therefore, a clear logical difference for Frege between the object-word "one" and the concept-word "one" or "unit", and we should not confuse the two. Indeed, he uses a different word in each case: "*Eins*" to denote the object and "*Einheit*" the concept. "Thus if we speak of 'units' [*Einheiten*]", he writes, "we cannot use this word to mean the same as the proper name 'one' [*Eins*] but we must use it as a concept-word" (GA 72 [§ 38]). There is only one "*Eins*", namely the number one, but many different things can have the property

of being an "*Einheit*", just as many can have the property of being a cat. The idea that "units" (*Einheiten*) are identical *objects* that are the components of numbers arises, in Frege's view, because we disregard this distinction.[16]

Frege describes what happens in that case as follows. We first treat the word "unit" (*Einheit*) as a concept-word and call the various objects to be counted "units" (as we might otherwise call them "cats"); we thus say that there are "many units" to be counted. Since, however, "unit" is here regarded as a concept-word, equivalent to "being a unit", the objects falling under the concept continue to be viewed as *distinct* things and so "difference is given its due" (GA 72 [§ 39]). We then convert the word "unit" into an object-word, and indeed conflate it with the word "one" (*Eins*); or, as Frege puts it, "the concept-word 'unit' transforms itself imperceptibly into the proper name 'one'" (GA 73 [§ 39]). We thus now take "unit" to stand for an *object*, not just a concept. Moreover, since we have said that there are "many units", we take the word "unit" – or "one" – to refer to *each* object before us. In this way, we come to think that we are confronted with many unit-objects, or "ones", all of which are identical to one another. This in turn leads us to think that the number of such objects is itself a set of identical units.

So, for example, we take the letters of the word "and" – "a", "n" and "d" – and bring them under the *concept* "unit" (*Einheit*); we thus think of each distinct letter as *being-a-unit*. We then (illegitimately) take each letter to be a unit-*object* that, as such, is identical with the others, and so we replace "a", "n" and "d" with "a unit and a unit and a unit" or "1 and 1 and 1". We thus see three identical units before us, and so we take the number 3 itself to comprise three such units. The popular conception of number is thus, for Frege, the result of an "artful manipulation of language", specifically of the word "unit" (*Einheit*), that remains hidden from us so long as we fail to distinguish carefully between concepts and objects (as well as between their corresponding words) (GA 73 [§ 39]).

So to pull Frege's thoughts together once again: he thinks that there cannot be many *ones* or *units* – that is, many identical objects – because they must either coalesce into one (and so no longer be many) or be different (and so no longer be identical ones or units). The popular idea that a number is a set of equal, featureless units is thus incoherent and unsustainable.[17] The thought that there *can* be many ones or units results from treating the word "unit" (*Einheit*) both as a concept-word *and* as an object-word. That thought can, however, be eliminated, if we adhere strictly to the distinction between concept and object.

Yet this distinction is presupposed by Frege without being derived from thought (or being) itself. It is indebted, ultimately, to the distinction between function and argument that Frege imports into his logic in order to render the latter congenial to arithmetic. Frege's solution to the problem he sees with the idea of there being "many ones" thus rests on what, to Hegelian eyes, is an

unjustified assumption. Furthermore, the problem itself presupposes Frege's distinctive conception of an object: for he denies that there can be unit-objects *equal* to one another, only because he holds an object to be singular, unique and so qualitatively *distinct* in some way from every other object. Like Leibniz, Frege takes it to be beyond question that "there are never two beings that are perfectly alike"; like Leibniz, therefore, he cannot countenance the thought of identical things: for "then there would be no *difference* between two *different* individuals".[18] Unlike Leibniz, however, Frege extends the latter's principle to the units that are traditionally thought to be contained in a number. For Frege, there cannot be identical units or "ones" even in a number, and so we need a new conception or definition of number altogether. His dissatisfaction with the traditional conception rests, however, on an assumption that he makes about all logical objects and that he never justifies.

It also rests, from a Hegelian point of view, on a failure to understand the logical structure of the one (*Eins*). To see this, we need to remind ourselves precisely how the one is to be conceived. This will first involve a short detour through the category of "something".

HEGEL ON "SOMETHING" AND THE "ONE"

The category of "object" (*Objekt*) does not arise in Hegel's *Logic* until the doctrine of the concept, which falls outside the scope of this study.[19] Nonetheless, a rudimentary object is derived in the doctrine of being, namely "something". As we saw in volume 1, chapter 8, this something and its counterpart, "the other", differ from one another in two ways. First, the other is simply that which is not the something itself: it is the "negative of something" (SL 90 / LS 111). So whatever "something" we take, that which is *other* than it is *not* it; to be "other" is thus to be different from the thing we start from. In this sense, Frege is right to say that "we call the object *another*, only because we can *distinguish* it from the first" (GA 69 [§ 36], emphasis added). Second, being "other" is not just being a "negation", but also being self-relating and having an identity of one's own. The mere negation of X is just that which is *not* X; that which is *other* than something, however, also occupies its own logical space and stands apart from the something. It is quite other than and separate from the latter, and as such is itself a separate something. What is other than something is thus not only different from it but also *something* else. As Hegel puts it, therefore, "something and other are, *first*, both determinate beings [*Daseiende*] or *somethings*" (SL 90 / LS 112).

The difference between being a mere negation and being an other is important but often overlooked. A mere negation, as Hegel understands it, cannot stand alone, but is always one side of a difference: it is the negation-*of*-a-reality. By contrast, whatever is *other* can stand alone, or rather apart, by

itself. What is other is, indeed, the counterpart of something, and so in that sense is not completely independent: one cannot be an other without something there for one to be other than. Nonetheless, what is other than something is precisely what stands apart from it – that which has a quite separate being of its own.

Note, however, that the other differs from the something to which it relates in both the respects we have outlined. It differs by *not* being that something and also by being *separate* from it. The two differences, however, have different logical consequences. Not being the something makes the other determinately or qualitatively different from that something: it makes it the "other" *rather than* the "something", that is, something *else*. Being separate from the something also makes the other different from the latter; yet separateness does not make the other qualitatively different, because it does not make it the explicit negation of the something. On the contrary, it turns the other into a something *just like* the first and so makes it *something* else (as opposed to something *else*).

The difference that lies specifically in separateness, or being self-relating, is thus an ambiguous one: for, as Leibniz would put it, it is a difference in which there is "no *difference*" between the items concerned.[20] It is, indeed, a *difference*, since the two items concerned are separate from one another; yet these two items are the *same* in having their own separate identities: they are both *equally* separate things. Such a difference is paradoxical, but it is one with which we are familiar from everyday experience: for we encounter it whenever I say "I" and you say "I" as well. Each of us is "I" and knows himself or herself to be a separate "I" with its own inner space. Equally, however, each of us knows that he or she is an "I" *just like* the other, that everyone is the *same* in being "I" or "me" (see PS 62 / 73).

There is a further ambiguity in the relation between something and other, for not only is each one something, but each is equally *other* than the other: either can be regarded as the first something, so either can be the other of that first (SL 90-1 / LS 112-13). In being other, however, each is *not* the something that its counterpart is; the qualitative difference between being "other" and being "something" is thus preserved, though it is reversible. By contrast, the ambiguity that lies in simple *separateness* does not preserve the qualitative difference between the two things. It reduces both to the same separate somethings.

Hegel insists that the difference between something and its other must include both qualitative difference (or negation) *and* separateness. Being "something or other" is a further form of determinate being (*Dasein*) and so is inseparable from negation; but being "something or other" is also being self-relating – being an "itself" – and so entails separateness. The two forms of difference are, however, in tension: for the moment of negation becomes explicit in the fact that something and its other are *other* than one another, whereas the moment of separateness is more explicit in the fact that both are equally *something*. In the

category of the *one* (*Eins*), by contrast, the moment of negation coincides with, and is submerged in, that of separateness and self-relation.

The one, as Hegel conceives it, is not just indeterminate being; it is determinate and so definitely itself, *not* anything else, and in this sense it has a limit. Yet it is also purely self-relating or pure "being-for-self". Accordingly, there is – initially at least – nothing else for it not to be (that is, nothing else is required by its logical structure). It is thus purely the *one* that it is and is not accompanied by anything other than it. Since it is determinate and so limited, and yet stands completely alone, the one, as Hegel puts it, is the "abstract limit *of itself*" (SL 132 / LS 166, emphasis added). It thus has its defining limit *in* its very self-relation, not in something outside it: it is definitely itself, and not anything else, in being the wholly self-sufficient one.

The idea that the one stands there purely by itself, and so *has no other*, distinguishes it from a mere something, which is always accompanied by something else. Logically, somethings always come in pairs, but the one is (initially) just the one that it is. The *qualitative difference* from something *else* does not, therefore, belong to the structure of the one, as it belongs to that of something. Strangely, this remains the case, even when there is more than just the one.

As we saw in volume 1, the one multiplies itself, through its own logic, into many ones (see 1: 262-6). This movement of self-multiplication is made necessary by the fact that the one contains a void within it. Briefly, what occurs is this. The one contains a void at its heart – it is empty and featureless – but, as the affirmatively self-relating *one*, it also differs from that void: it is a further form of being, rather than of nothing. Yet since it is the one, it cannot just be the negation of, and qualitatively different from, the void; it can differ from the latter only by being purely self-relating, purely *for itself*. That is to say, it can differ from the void only by *separating* itself from it altogether. By separating itself from the void, however, it separates the latter in turn from it, and so expels the void from itself (SL 133-4 / LS 169). Through its own logic, therefore, the empty one *excludes* its own void and so mutates into a one hanging in a void that lies outside it. Yet since that void is now *separate* from the one, it must itself be another empty self-relating being, that is, another empty *one*. The one, therefore, does not just stand alone, as we first thought, or just hang in the void, but it must be accompanied by *another one*. That one, however, must expel its void in turn and set the latter outside it as yet another one; and so on, and so on. "The one is consequently *the becoming of many ones*" (SL / 136 / LS 171).

It is not my intention here to trace once again the complete logical development of the one. All I wish to show is that the nature of the one makes necessary a difference between entities that is itself different from qualitative, determinate difference. There is a *qualitative* difference between something and its other, between being-in-itself and being other-related, and between

determination and constitution. In each case this qualitative difference consists in the fact that the categories concerned are *not the same* as one another. The categories are not just different but also form a unity (in the case of being-in-itself and being other-related) or turn dialectically into one another (as in the case of determination and constitution); yet the qualitative *difference* between them remains. As Hegel writes, "in the field of the qualitative, differences in their sublatedness also retain immediate qualitative being relative to one another" (SL 96 / LS 119).

Something and its other are thus not only the same, but also definitely different. This in turn provides the logical ground for Frege's assertion that "two objects are *never* thoroughly equal" (GA 66 [§ 34]). From a Hegelian point of view, Frege exaggerates the distinct, *singular* character of objects because he overlooks the fact that *universal* categories are constitutive of their being singular objects (and are not just concepts "under" which such objects "fall").[21] Objects, for Hegel, are not purely singular, but also identical to one another in various respects, because – besides sharing properties, such as "red" – they are all *somethings*, all *objects*. Nonetheless, Frege is not wrong to claim that there are qualitative differences between them.

What Frege ignores, however, is that the one does not, and cannot, differ from another one in this qualitative way. The one does, indeed, give rise to many other ones. Yet these ones are not qualitatively *other* than one another; on the contrary, each is the *same* empty one. Something differs from its *other*, but the one (*Eins*) differs from nothing but another identical *one*. The difference between the ones does not, therefore, involve their being the direct negation of one another: it does not consist in the one's *not* being what the other one *is*. It consists simply in each being a *separate one* for itself.

This difference that – unlike that between something and other – is not at all qualitative is made necessary by the logical structure of the one. The one is a definite, determinate entity, not just indeterminate being; but its distinctive character lies in being purely for itself. As a one, therefore, it is not the negation of something else, but it is purely *self*-relating. Since the one consists in being for itself, in standing alone, and not in *not*-being-something-else, the only way it can differ from other ones is by standing apart and being separate from them – and this is true of every one. Each one thus differs from the others by repelling them and holding them at bay, not by having a different logical structure; indeed, all have precisely the same structure, since they are all purely self-relating and empty in the same way. The one differs from all other ones, therefore, solely by being separate, *just as they are*. All ones, or units, are thus equal to one another, since each is simply a separate, self-relating and utterly featureless entity.

In the *Foundations* Frege asks "*are units equal to one another?*" (GA 66 [§ 34]) and his answer, as we have seen, is no. He gives this negative answer because

he assumes that all objects differ in a way that *prevents* them from being identical (though they can "fall under" the same concept, including that of "being-an-object"). For Frege, if two objects are judged to be equal or identical to one another, they are in fact one and the same thing and not *two* objects at all; for them to be two objects, they have to be explicitly different from one another: two distinct, singular things. From a Hegelian viewpoint, this shows that Frege equates the difference between objects with *qualitative* difference, with being this-*not*-that. He fails to recognize, however, that objects can also differ in a different way, namely by simply being separate or "for themselves", and that units or "ones" not only can but *must* differ in this way. This failure in turn stems from his failure to understand the distinctive logical structure of the one. Leibniz would accuse Frege of failing to recognize the distinctive character of "numerical" difference (see 2: 94). For Hegel, however, such difference is itself made necessary by the structure of the one. Frege's more significant failure, therefore, is that he does not understand the logic of the one (*Eins*) properly.

Frege begins from the assumption that there is a sharp difference between the words "one" (*Eins*) and "unit" (*Einheit*). The word "one" designates a specific object, namely the number one, and, for Frege, there is only one such number (GA 71 [§ 38]). The word "unit", by contrast, is not an object-word but designates a concept under which objects can fall. Distinct objects can thus have the property of being units, but there cannot be any unit-*objects*, and so there cannot be any objects that, *as objects*, are completely equal to one another. Furthermore, if anyone thought there could be, the objects would just coalesce into one, since their very equality would prevent them from being different objects (see GA 75 [§ 41]).

From a Hegelian viewpoint, however, Frege's guiding assumption is merely that – an *assumption* – and if we philosophize without such assumptions, we come to a different understanding of "one" from Frege's. In particular, we discover that being *one* is significantly different from being *something*. Like "being something", "being one" is not just a concept "under" which presupposed objects "fall", but it constitutes its object. So just as there must be at least two somethings in Hegel's world, there must also be self-sufficient ones (or units). To be one, it should be noted, is not yet to be the *number* one, but it is the fullest realization of quality: it is to be "oneself" to the highest degree. Unlike being something, however, being one also represents the *loss* of qualitative distinctness: for while something is bound to an other, the one has no qualitative other. The one is initially all alone in its own space or "void", and then it is surrounded by other ones *just like it*. Quality thus makes necessary many ones, each of which is just as much one as the others and between which *there is no qualitative difference*. The only difference between them is separateness or "repulsion": the difference between ones that are all *equally* separate. *Pace* Frege, therefore, not only can there be unit-objects equal to one another, but there must be.[22]

Being one eventually makes qualitative difference necessary again, namely between repulsion and attraction, but the difference between ones themselves remains non-qualitative. These separate-but-equal ones then form themselves into *quantity* and their separateness becomes the quantitative moment of discreteness. The equality or sameness of such units in turn constitutes the moment of continuity. So although equal units do not just collapse into "one thing", as Frege claims (GA 68 [§ 35]), because they remain differentiated by their separateness, they do form a single *continuity* or unity in their separateness.

Quantity, as we have seen, then gives rise to the quantum, which is a limited portion of quantity and thus a limited unity of discrete units. When this quantum is understood to contain a specific amount (*Anzahl*) of such units, it is conceived as a number (*Zahl*) (SL 168-9 / LS 213-14). Number is thus the very thing that Frege claims it cannot be, namely a unified plurality of *discrete but equal* units; and it must contain such discrete, equal units because it is made necessary by the logic of the one. Frege's assumptions about objects and concepts, and "*Eins*" and "*Einheit*", turn him against the popular conception of number, therefore, because they prevent him from understanding the logical structure of the one (*Eins*) as such.

FREGE ON NUMBERS, OBJECTS AND CONCEPTS

Dummett thinks that Frege provides a brilliant and definitive critique of the idea that a number is a "set of featureless units".[23] We have seen, however, that – from Hegel's perspective – that critique is far from definitive, since it rests on unwarranted assumptions and ignorance of the true nature of the one. As we shall now see, Frege's own positive conception of number also rests on his distinctive assumptions.

Frege notes that numbers often look as if they are properties of *things*: we talk, for example, of the "number *of bales*" just as we talk of the "weight *of bales*" (GA 86 [§ 52], emphasis added). This appearance is reinforced by the fact that number words, such as "three" and "four", are frequently treated as adjectives, like "the words hard, heavy, red", which clearly "designate the properties of external things" (GA 50 [§ 21]). So in the expression "four thoroughbred horses" the word "four" is taken to qualify "thoroughbred horse" just as "thoroughbred" qualifies "horse" (GA 86 [§ 52]).

Frege insists, however, that numbers are very different from properties of things. When, for example, we say that a tree has green leaves, we attribute the property "green" to each of the leaves in turn, but we do not do this when we say that the tree has 1,000 leaves: clearly, each leaf is not "1,000" in the way that it is "green". The number "1,000" is thus not a property of the leaves as "green" is. Furthermore, if I change the way I consider the colour of something, that colour does not itself change: if I focus on part of a red thing, or think of

it as made up of several red segments, its colour remains red. By contrast, if I change the way I consider the number belonging to something, that number can change: so "I can consider the *Iliad* to be one poem, or 24 books, or a great number of verses" (GA 51 [§ 22]). The number I associate with the *Iliad* is objective, since there is, indeed, *one* poem that comprises *24* books;[24] but the specific number we attach depends on how we choose to regard it in a way that the colour of my copy of the book does not.

Despite initial appearances, therefore, a number, in Frege's view, is not a property of a thing like its colour, weight or texture. This has two related consequences, both of which are made necessary by Frege's sharp distinction between a concept and an object.

First, recall that, for Frege, a property of an object is equivalent to a concept under which the object falls (see FR 189-90 / ÜBG 56, and 2: 44). Since a number is not such a property, it cannot be a *concept* as Frege understands it. That means, however, that it must be an *object*. This in turn is indicated by the fact that the definite article precedes every number word: we speak of *the* number one, *the* number two, and so on (GA 71 [§ 38]). Number words are, indeed, often used "adjectivally" or "attributively", and so create the illusion that numbers are properties of things (or concepts under which the things fall) (GA 50 [§ 21]. This illusion can, however, be dispelled by recasting the sentences in which number words occur: "for example, the proposition 'Jupiter has *four moons*' can be converted into '*the* number of Jupiter's moons is four'" or, indeed, into "*the* number of Jupiter's moons is *the* number 4" (GA 90 [§ 57], emphasis added). In the original sentence the word "four" is treated as an adjective, like "red", and so as a property- or concept-word; but in the second of the two new sentences "four" (or "4") is clearly part of an object-expression that designates the number four.

Second, if a number is not a concept, but an object, then it is not itself that under which *objects* fall. Numbers, therefore, do not qualify objects, and a statement assigning a number to something is not a statement *about any object*. This seems counter-intuitive, for, as Frege points out, "language ascribes number to objects" (GA 85-6 [§ 52]); indeed, it does so even when we recast sentences in the manner indicated in the previous paragraph. If we say "Jupiter has four moons", and so treat "four" as an adjective, then "four" evidently appears to qualify objects, namely the (Galilean) moons of Jupiter. This remains the case even when we revise the sentence and say "the number of Jupiter's moons is the number four". It is clear from this sentence that four is an object, not a property, but it seems equally clear that four is an object that belongs in some way to other objects, namely (once again) Jupiter's moons. We seem to be saying that the number four belongs to Io, Europa, Ganymede and Callisto taken together, whereas the number two belongs to the pairing of Io and Europa alone. For Frege, however, a number cannot be ascribed to objects, precisely

because it is an object itself. One object can be identified with another – we can recognize that they are in fact the same object, and not two different objects after all – and one can form part of another, but an object cannot belong or be ascribed to another (or to a set of others); only a *property* can belong to an object. A property belongs to, and so qualifies, an object insofar as the latter falls under the equivalent concept; yet a number is neither a property, nor a concept, but an object; it cannot, therefore, be ascribed to an object itself.

Yet if statements of number are not directly about objects, what are they about? The sharp distinction between object and concept points to Frege's famous answer: "a statement of number contains an assertion about a concept" (GA 81 [§ 46]). Thus, when we say that the number of Jupiter's moons is four, we are assigning the number four, not to certain objects orbiting Jupiter, but to the concept "moon of Jupiter". We can see that the phrase "moon of Jupiter" is a concept-expression, rather than an object-expression, because it is capable of being given a plural form. For Frege, as we know, an object-expression is always singular because it takes the form "*the* such and such" (or is a proper name): an object does not occur in multiple forms of itself, but is just *the* object it is (see GA 85 [§ 51]). "A plural", as Frege states, is thus "possible only of concept-words" (GA 72 [§ 38]): we can talk of a moon of Jupiter, but also of several such moons. Accordingly, the very fact that we can say that the number of Jupiter's *moons* is four shows that we are assigning the number to a concept.

Since a number is an independent object, however, it cannot be described as a "property" of the concept to which it ascribed (GA 90 [§ 57]). Nonetheless, a property is attributed to a concept whenever a number is ascribed to the latter. This is the property of having objects fall under it. When we say that the number of Jupiter's moons is four, therefore, we are not saying something directly about a set of objects in space; rather, we are saying *of the concept* "Jupiter's moon" – that is, of the concept "( ) is a moon of Jupiter" – that objects fall under it in a certain way that differs from that associated with the numbers 3 or 5.

Note, by the way, that we cannot simply say that Jupiter has four moons when the concept "( ) is a moon of Jupiter" has *four objects* falling under it, since we would then take the number four to apply to objects, rather than just concepts, after all. Frege's distinction between concept and object is absolute; so if a number applies to concepts, it does not apply to *objects* (and, equally, if it is itself an object, it cannot be a second-level *concept* under which first-level concepts fall). Frege has thus to find a way of explaining what it means for a number to belong to a concept *without* applying that number to the objects that fall under the concept. He begins to do this in § 55 of the *Foundations*.

Frege starts with the number 0 and says first that "the number 0 belongs to a concept if no object [*kein Gegenstand*] falls under it" (GA 88 [§ 55]). He then immediately points out that the word "no" (*kein*) is actually just another word

for "0"; so he is in fact saying that 0 belongs to a concept if 0 objects fall under it, which is precisely the statement he has to avoid (since it assigns the number 0 to objects). This problem can be solved, however, if we eliminate the word "no" and recast our statement as follows: "the number 0 belongs to a concept, if, whatever *a* may be, the proposition holds universally that *a* does not fall under that concept" (GA 88 [§ 55]).

Two things should be noted about this sentence. First, the sentence explains what it means for a number to belong to a concept via the idea that objects fall – or do not fall – "under" that concept, but it does not assign a number to those objects themselves. Specifically, it states that 0 belongs to a concept if, whatever the designated object is, that object does *not* fall under the concept. The sentence thus eliminates the idea of "0 objects" and assigns 0 to a concept alone.

Second, Frege's sentence makes use of the quantifier that he introduced in the *Begriffsschrift* and so has what he considers to be the proper logical form. It says: the number 0 belongs to a concept if, "whatever *a* may be" – that is, *for any a* – a does not fall under the concept. Frege thus explains what it is for a number to belong to a concept in purely *logical* terms, employing only the quantifier, the conditional, negation, the form of a singular proposition and the idea of an object falling, or rather not falling, under a concept. This does not yet define a number itself in purely logical terms, but it is a step in that direction: for we can now make purely logical sense of the idea that a number (namely 0) belongs to a *concept*, rather than objects – and especially rather than *unit*-objects, as the traditional view would have it.

After considering 0 Frege turns his attention to the number 1. It is, of course, tempting to say that 1 belongs to a concept when 1 object falls under it; but this would do the very thing that ascribing a number to a concept is meant to exclude, namely assign the number 1 to an object. So Frege formulates the following, purely logical, claim:

> the number 1 belongs to a concept *F*, if, whatever *a* may be, the proposition does not hold universally that *a* does not fall under *F*, and if from the propositions "*a* falls under *F*" and "*b* falls under *F*", it follows universally that *a* and *b* are the same.
>
> —GA 88 [§ 55]

So 1 belongs to a concept under two conditions. On the one hand, 0 does not belong to the concept; that is to say, it is *not* the case, that for any a, a does *not* fall under the concept; or, put positively, an a *does* fall under the concept. On the other hand, for any a, if this a falls under the concept and anything other than this a falls under the concept too, that other – let's call it b – is in fact the same thing as this a. So 1 belongs to a concept if, for any a, this a, and this a

alone, falls under the concept. All of this, of course, is a roundabout way of saying that there is 1 "moon of Jupiter" or 1 "queen of England" if just *1 object* falls under either concept. Yet Frege's sharp distinction between concept and object requires him to avoid assigning a number to the object falling under the concept to which he assigns the number, so he has no choice but to adopt his convoluted, but logical (in his sense), mode of expression.

Frege concludes § 55 by proposing an explanation of what it means for any number that follows another one to belong to a concept. He constructs the variable (n + 1) to stand for such a number; so if n is 0, the number will be 1, and if n is 1 the number will be 2, and so on. Frege's explanation is as follows:

> The number (*n* + 1) belongs to the concept *F* if there is an object *a* falling under *F* such that the number *n* belongs to the concept "falling under *F*, but not *a*".
>
> —GA 89 [§ 55]

Again, two conditions must be met for the number (n + 1) to belong to a concept F. On the one hand, an object a must fall under the concept F. On the other hand, the number n must belong to the concept "falling under F, but not a". Any object falling under this latter concept, however, also falls – *besides* a – under the concept F. The number n thus indicates the objects that fall under concept F besides a, with the consequence that, when a is also taken into account, the number of the concept is (n + 1). To find out what it means for a specific number (n + 1) to belong to a concept, therefore, we simply have to identify n with a known number.

So far we have two options to choose from, since we know what it means both for 0 and for 1 to belong to a concept. So let us start first with n = 0. In this case, the number (n + 1) that belongs to the concept F is 1, since *a* falls under F and 0 is the number *n* belonging to the concept "falling under F, but not a". 0 is in turn this latter number (following what was said above), if, for any *b*, b does *not* fall under the concept "falling under F, but not a". If, however, whatever b is, it does not fall under this latter concept, then it cannot fall – next to a – under the concept F either; that is to say, nothing but *a* falls under F. The number 1 belongs to concept F, therefore, if a falls under it *but nothing else does*.[25]

We can now use this idea to explain what it means for "n + 1" to belong to a concept when n = 1, that is, what it means for the number 2 to belong to concept F. This number belongs to the concept F when a falls under the concept and the number 1 belongs to the concept "falling under F, but not a". The number 1 belongs to this latter concept – let's call it F1 – when (changing the variable) x falls under it but nothing else does. If x alone falls under F1, however, it is the only thing that falls under F and is not a; so only x and a fall under F.

"1 + 1", or 2, belongs to concept F, therefore, when x and a fall under it *but nothing else does*. In this way, we can proceed to explain what it means for any number to belong to a concept; and we can do so using only the resources of Frege's logic and without assigning a number to any object.

For Frege, two advantages (at least) are to be gained from ascribing numbers to concepts rather than objects. First, we can make sense of the number 0, which, we recall, Frege takes from the start to be a number like any other. If we think of numbers as belonging to objects, and say, for example, that "Venus has 0 moons", then 0 poses a problem: for by definition there are no objects – in this case, no moons – to which it can belong. On Frege's understanding, however, the problem is removed: for it is not objects but "the *concept* 'moon of Venus' to which a property is ascribed, namely that of including nothing under it" (GA 81 [§ 46]). Second, we can explain how *different* numbers can appear to be assigned to the same object, for example, the *Iliad*. Assigning different numbers to the same object would be unobjectionable if numbers were subjective; but Frege insists that they are objective (GA 57 [§ 26]), so assigning different numbers to the same thing looks contradictory (since 1 object is also said to be e.g. 24 objects). Contradiction is avoided, however, if we regard concepts, rather than objects, as bearers of numbers: for we can now say that the number 1 belongs to the concept "being the *Iliad*" because the *Iliad* alone falls under that concept, whereas 24 belongs to the concept "being a book of the *Iliad*" because books 1, 2, 3 and so on of the *Iliad* all fall under it. In this way, while we appear to assign different numbers to the same object, we avoid contradiction because we actually assign them to different concepts.[26]

Frege's logical distinction between objects and concepts is thus central to his conception of numbers. In the *Foundations* Frege does not discuss the ascription of numbers to relations, rather than concepts, as in "Jupiter has 70 *more* moons than the Earth". Yet this does not leave it open to Frege to assign numbers to objects after all: for relations are just as distinct from objects as concepts are.[27] Frege's main point thus remains unaffected: numbers are not assigned to objects. Numbers do not, therefore, qualify objects directly, as adjectives do; and, as we saw above, they are not themselves sets of featureless unit-objects. From a Hegelian point of view, however, Frege's conception of numbers is problematic.

The relation between "things" and their "properties" is not examined by Hegel until the doctrine of essence. Yet what proves to be a property in the sphere of essence has its equivalent in the sphere of being, namely in the immediate *quality* of something. Note, however, that a quality is not attached by Hegel to a thing that is distinct from it and precedes it logically. Being first proves to be determinate being and quality, and then the latter in turn takes the form of something (or a rudimentary object). A something is thus self-relating quality – quality that forms itself into a "this" – rather than a "this" to which a

quality "belongs". In this sense, Hegel's metaphysics resembles that of Spinoza, for whom being or "substance" individuates itself *into* finite modes, and it is at odds with Frege's ontology in which individual objects are logically "primitive" and given from the start.

Since a something is *quality* in the form of a "this", rather than a *this* that "has" qualities, quality is itself constitutive of something. There can thus be no sharp distinction between something and its qualities of the kind that Frege postulates between objects and their properties (or the concepts under which they fall). From a Hegelian viewpoint, therefore, a number cannot be neatly assigned to a property-cum-concept, *as opposed* to an object. If it is to be assigned to a property-cum-concept, it must be assigned to an object or objects, too, since properties, in the form of qualities, constitute objects: something has its being in and through its qualities, "properties" or "concepts". (So if 24 is assigned to "book of the *Iliad*", it must equally be assigned to the *books* of the *Iliad* – taken together, of course, not individually. And if 1 is assigned to "being the *Iliad*", it must also be assigned to the *Iliad* itself – which in this respect is, logically, a different object from the books it contains.)

As I have noted above, things are also constituted by the logical *categories* that give them their form (see 2: 83). The categories of being-in-itself, being-for-other, limit and finitude are thus not just concepts "under" which things "fall", but they are constitutive *of* such things. The same is true of the "concept" in the third book of the *Logic*: it too does not just subsume previously given objects under it, but it structures objects themselves (namely, insofar as they are mechanical, chemical or organic). So again, if numbers are assigned to concepts, as Frege has it, they must also be assigned to the objects that the concepts constitute, since the two are inseparable. This is true whether the equivalent of Frege's "concept" in Hegel's thought is taken to be "quality", "category" or "concept" (in its *begriffslogisch* sense).

Hegel also has a more direct reason for thinking that numbers belong to things. Number, for him, is a form of quantity, and quantity itself is a form of *being*, made necessary by quality; since quantity is inherent in being, and being individuates itself into beings or "somethings", quantity and number must both belong to the latter (if not to everything equally or in the same way). Having said this, Hegel would agree with Frege that a number is different from a property or quality of a thing, such as "red". After all, quantity, for Hegel, is logically distinct from quality, even though it is made necessary by the latter, and number is a determinate unit of quantity. A number thus cannot constitute the being of something in the intimate way a quality does. The logical difference between quantity and quality is evident in the fact that a thing can change its quantity and number, and so become bigger or smaller, without ceasing to be what it is, but it cannot change its defining quality in the same way and remain itself. A field can become bigger and remain a *field*, but if it changes the

quality that makes it a field in the first place, it will become a wood or a marsh (SL 153 / LS 193).

Now what interests Frege is not that a thing can change its own quantity and with it the number that belongs to it, but rather that *we* can change the number we assign to a thing (or, rather, that we appear to assign to a *thing*). As we saw above, we can regard the *Iliad* as 1 poem or as a collection of 24 books, and in each case our statement of number is objective; we cannot, however, alter the colour or hardness of a thing simply by changing the way we think about it (GA 51-2 [§ 22]). Frege thus highlights a different aspect of number from Hegel. Nonetheless, both philosophers recognize that numbers are importantly different from the immediate qualities of things.[28]

In Hegel's view, however, this difference does not require us to ascribe numbers to concepts, rather than things. Number – and quantity more generally – remains a constitutive (though changeable) feature of *things*: it belongs to what a thing itself *is*. Later in Hegel's system forms of being will emerge – namely organic and self-conscious being – in which quantity and number play a less important role than they do in inorganic nature (see EL 159-60 / 212 [§ 99 A]); and, as I noted in the last chapter, the dialectical character of the *categories* of the *Logic* means that they can be counted only if a certain violence or "madness" is inflicted on them.[29] Nonetheless, the being that is thought through, and structured by, those categories proves to *be* quantitative and to *be* numerable itself. Quantity and number must belong to beings, therefore, even if the latter can change their particular quantity and number without ceasing to be what they are (and in that sense are "indifferent" to them) in a way they cannot change their defining quality.[30] Indeed, in the sphere of measure quantity and number prove to be tied even more closely to things: for there we see that certain quantities and numbers are not just a matter of indifference to things, but mark the limits within which those things must remain if they are to preserve their defining qualities and be what they are.

Before we proceed, however, we need to qualify slightly what we have just been saying. Speculative logic shows that being involves various forms of qualitative being: being determinate, being something, being finite and being one. Logic also shows that qualitative being gives rise to quantity, via the category of the one. Quantity, however, is *different* logically from quality, and Hegel explains this difference in the first remark on quantity in the manner described above: something can change its quantity without ceasing to be itself (disregarding its measure, which has not yet been derived), but it cannot change its defining quality and remain what it is in the same way. Yet strictly speaking, at the point we have now reached in the *Logic* – the account of number – Hegel has not yet *proven* this claim from the first remark. This is because he has not yet proven that quantity *coexists* with quality at all: he has shown only that it arises from quality and displaces the latter. (Quantity displaces quality because

it is generated by the one [*Eins*] in which qualitative difference already disappears.)[31] Later in his account of quantity, however, Hegel will show that quantity and quality do in fact coexist, but that quality remains indifferent to the quantity that belongs to it. At that point (in 1.2.2.B.b), therefore, it will become clear that quality remains as its quantity changes, and so the claim made in the remark will be demonstrated. Hegel proves this claim by showing that quantity does not just exist on its own, but proves to be the quantity of *something* (*Etwas*). Prior to this point, quantity divides itself into quantitative "somethings" or quanta, but it does not show itself to be the quantity of a *qualitative* something; this occurs only in 1.2.2.B.b. It is only at that point, therefore, and not before, that quantity and number demonstrably belong to *things* (that are not themselves just quanta). We asserted above that number must belong to things because quantity and number are inherent in being and being individuates itself into things. This assertion remains true; but it receives its proper justification, only when we see being give rise not just to qualitative somethings, and not just to quanta, but *to quanta that belong to qualitative things*. As we shall see, this occurs when we consider the relation between extensive and intensive magnitude.[32]

For the purposes of comparing Hegel with Frege, therefore, I am anticipating here what will be proven only later: for Hegel, in contrast to Frege, number belongs to *things*, even though he agrees with Frege that number is not a *quality* of those things, like "red" or "hard". This does not mean, by the way, that Hegel would reject out of hand Frege's claim that numbers are ascribed to *concepts*; or, rather, it does not mean he would reject the claim that they *can* be ascribed to concepts, as well as to things. To see this, however, we need to substitute a different term for "concept". Frege equates "concepts" with the "properties" of things, and in Hegel's doctrine of being "properties" appear in the form of *qualities*. What we need to consider, therefore, is whether Hegel might accept the idea that a number can apply to things with a *shared quality*.

For Hegel, a number is a unified set of empty units, each of which is the same as the others. (Frege tries to reject this conception of number, but, as we saw above, his critique misses its target.) In order to be counted, therefore, things must be treated as, and must in one respect *be*, identical empty units: they must be the *same*. It is perfectly compatible with this idea, however, that they all be the same in some qualitative respect, too – that they share a common quality. They can, of course, simply be regarded as units or bare objects; but they can also be regarded as units of sand, or as apples or pears, and still be counted as units.

The idea that objects *must* share a common quality in order to be counted is found in Aristotle (see 2: 22-3). In the *Metaphysics*, for example, Aristotle asserts that things must have a common "measure", and so all be of *one* kind, if they are to be counted: "the measure must always be something predicable of all alike, e.g. if the things are horses, the measure is a horse, and if they are men,

man. If they are a man, a horse, and a god, the measure is perhaps living beings, and the number of them will be a number of living beings".[33] Jakob Klein argues that, for Aristotle, this measure alone gives unity to a number: a number is never just a unified set of bare units, but is always the set of *these* things, rather than those, regarded *as* units.[34] In this respect, therefore, Aristotle and Hegel are clearly at odds with one another, since Hegel thinks that a number for itself is, indeed, a unified set of bare units. Hegel would thus reject the Aristotelian idea that objects *must* share the same quality (beyond just being units) in order to be counted. Nothing in what he says, however, excludes the possibility that countable objects, as well as being units, *can* also share a quality.

Note that in the cases that Aristotle has in mind, a number is associated directly with the *quality* shared by the things: there are 2 "horses", 3 "men", and so on. In Frege's terminology, the number is thus ascribed to the "concept" concerned. Yet the number is also associated with the *things* that have that quality, with the horses and men themselves. Aristotle is explicit about this: "number and magnitudes cannot exist apart from things".[35] So, just because a number is ascribed to a "concept" – that is, to a quality or "measure" shared by things – this does not mean that it cannot also be ascribed to things. Indeed, on the view suggested by Aristotle's remarks, number must be associated with both at the same time. This would be the Hegelian way of looking at the matter, too (if countable things are to share a quality): for Hegel thinks that qualities constitute the *things* to which they belong.

To repeat: for Hegel, the ascription of a number to things does not require reference to a common quality, since the things can be regarded as, and in one respect are, bare units. This, however, does not preclude ascribing a number to things insofar as they have a common quality, too. In that case, in Frege's terminology, the number belongs both to the "concept" under which the things "fall" *and* to the things themselves. The one does not exclude the other.

A similar thought to Aristotle's is found in Spinoza's letter 50 (to Jarig Jelles). Spinoza states that "we do not conceive things under the category of numbers unless they are included in a common class [*genus*]. For example, he who holds in his hand a penny and a dollar will not think of the number two unless he can apply a common name to this penny and dollar, that is, pieces of money or coins".[36] Once again, then, for objects to be countable they must have some property or quality in common. This passage is cited by Frege in the *Foundations* in support of his idea that numbers are ascribed to concepts, rather than objects (GA 83 [§ 49]). There are two ways, however, in which Frege misunderstands the import of this passage. First, Spinoza himself thinks that number is simply a "mode of imagining" (partly because we can delimit quantity, and so assign it a number, "as we please"). He thus takes number to be *subjective* in a way that would not satisfy Frege.[37] Second, Spinoza maintains nonetheless that we "conceive *things* [*res*] under the category of numbers" (emphasis added). It is

clear, therefore, that, for Spinoza – despite the fact that numbers are ultimately subjective – nothing in the thought that objects must have some quality or "name" in common in order to be counted means that numbers cannot also be assigned to those *objects*. The fact that a penny and a dollar have both to be regarded as "coins" in order to be counted does not mean that, strictly speaking, we should not think of them as *two* coin-*objects*. "Two" is not a *quality* of objects like being "a coin" or being "a penny"; nonetheless, this number can be assigned to *objects* when they come in pairs, and so need not just be ascribed to the "concept" under which they "fall".

These thoughts, and others set out above, show therefore that there is no good reason why Frege cannot say in the *Foundations* that the number 1 belongs to a concept because 1 *object* falls under it (though there are good reasons for dropping the idea of objects "falling under" concepts); and that in turn means that there is no good reason for him to resort to his convoluted logical formulations to try to avoid saying this. The number "0", by the way – assuming for the moment that it is a number just like 2, 3 or 4 – poses no special problem for this way of regarding numbers: for, as Frege himself notes before rejecting the formulation, he can simply say that "the number 0 belongs to a concept if *no* object falls under it" (GA 88 [§ 55], emphasis added).

So to sum up: from a Hegelian point of view, Frege goes wrong in three ways. First, he fails to see that a number can – indeed, must – be a unified set of empty units or ones, because he fails to understand the distinctive logical structure of the "one" (*Eins*) as such and the way in which it generates multiple instances of itself. This in turn is due to his dogmatic insistence that all objects must be qualitatively different from one another. Second, he fails to see that numbers belong to *objects* or *things* directly and that objects can thus be counted as bare units without a common quality or "concept" (beyond the quality that makes them "units"). Third, he fails to see that, when countable objects do have a common quality or "concept", this does not mean that number belongs to that concept alone and *not* to the objects themselves. Frege's failure in this regard is due to his dogmatic insistence that concepts and objects, though bound together by the unsaturated nature of the former,[38] are logically distinct, indeed opposed to one another. As we shall now see, this sharp distinction between concept and object continues to govern Frege's thinking as he endeavours to *define* number as such.

FREGE'S DEFINITION OF NUMBER

The criterion of identity for numbers

Frege's overall project is to prove the laws of arithmetic, as well as mathematical equations, to be analytic by deriving them from the laws of logic and the

definitions of number and of individual numbers.[39] In § 56 of the *Foundations*, however, Frege insists that he has not yet provided the required definitions. All he has done, he tells us, is explain what it means for a number to "belong to" something. First, he has argued that a number belongs to a *concept*, rather than to objects; second, he has explained, in purely logical terms, what it means for 0, 1 and "n + 1" – and thus any natural number – to belong to a concept. Yet this, he claims, does not amount to *defining* 0, 1 or any other number. As he puts it, "it is only an illusion that we have explained [*erklärt*] 0 and 1; in truth we have only determined the sense of the phrases 'the number 0 belongs to [*kommt zu*]', 'the number 1 belongs to'" (GA 89 [§ 56]).

As a consequence, in Frege's view, we do not yet know precisely *what* 0, 1, 2 or any other number is: we do not know what makes each one the specific "reidentifiable" (*wiedererkennbar*) object it is and so distinguishes it from every other number. This in turn means that we have no way of proving that two expressions standing for the same number do in fact stand for the *very same* number: we cannot prove "that if the number *a* belongs to the concept *F* and the number *b* belongs to the same concept, then necessarily *a* = *b*". Moreover, we do not know what a number as such is, and so what makes 0, 1 and 2 numbers at all. This means, however, that at this point we cannot exclude the possibility that objects we do not usually consider to be numbers are in fact numbers. So, to use what Frege himself calls "an extreme example", we cannot exclude the possibility that "Julius Caesar" is a number: we cannot decide "whether the number *Julius Caesar* belongs to a concept, or whether that well-known conqueror of Gaul is a number or not" (GA 89 [§ 56]). Frege's next task, therefore, is to define the term "number" (*Anzahl*), and then, on that basis, to define the terms that denote the individual numbers (*Zahlen*), starting with "0" and then "1", and so on. We shall now consider how Frege carries out this task.

In § 62 of the *Foundations* Frege begins by asking how a number can be "given" to us, since we have no sensuous "representation" or "intuition" of numbers (since they are purely logical objects). In response to this question he then invokes the "context principle" that he introduced earlier in the book: "only in the context of a proposition [*Satz*] do words mean something" (GA 94 [§ 62]; see 23 [Intro.]). The implication is that numbers are "given" to us through the way in which number *words* are used and understood in sentences. We need to examine the latter, therefore, if we are to comprehend fully, and to define, what a number is; or, as Frege writes, everything depends "on explaining the sense of a proposition in which a number word occurs".

With this suggestion, Dummett argues, Frege carries out the "linguistic turn": for he poses a "*non*-linguistic question" – "what is a number?" – and returns a "linguistic answer". In so doing, we are told, Frege proves to be "the grandfather of analytical philosophy"; indeed Dummett asserts, with

breathtaking parochialism, that "§62 is arguably the most pregnant philosophical paragraph ever written".[40] By Dummett's own admission, however, Frege provides no explanation or support for his linguistic turn: "he offers no justification for making it, considers no objection to it and essays no defence of it: he simply executes the manoeuvre as if there were no novelty to it".[41] Dummett is untroubled by this, but Hegel would regard it as another example of question-begging by Frege. We saw above that Frege's account of number is determined by the distinction he takes for granted between concepts and object, and now we see once again that it rests on a presupposition – the context principle – that is never derived and justified and never called into question.

Having made his linguistic turn, Frege then narrows the range of sentences we are to consider if we are to understand number properly. After all, we often use number words in what Frege considers to be a misleading way: for example, we use them "adjectivally" and so wrongly conclude that numbers are properties of things, like "red" (see GA 50 [§ 21]). We thus need to identify those sentences that reflect, and so will enable us further to comprehend, the distinctive character of numbers. According to Frege, numbers are "independent objects" (GA 94 [§ 62]) and, as such (as he states in a later essay), each one is "equal to itself" (*sich selbst gleich*) (FR 143 / FB 16). That in turn means that each one must be capable of being recognized as the *same again* when it appears "in another guise" (GA 98 [§ 66]). The sentences we must consider, therefore, are precisely those that express "recognition" of a number as the same again (*ein Wiedererkennen*) (GA 94 [§ 62]). Now a number, for Frege, belongs to a concept, not to objects. The specific sentence we must consider is thus the following: "the number that belongs to the concept *F* is *the same* [*dieselbe*] as the number that belongs to the concept *G*".

By itself, however, this sentence merely tells us *that* one number is the same as another. It says nothing about the particular circumstances under which numbers, as opposed to other objects, are recognized as the same. It thus cannot help us comprehend what is distinctive about numbers after all. So how are we now to proceed? The answer is as follows. Earlier in the *Foundations*, Frege maintained that no one has yet been able to say precisely "what number is" (GA 16 [Intro.]); nonetheless, "number", for him, is not a completely empty word, but one that is familiar to us. Our task now, therefore, is to identify when and how numbers, as we usually conceive them, are understood to be the same or equal, and then to draw on this insight to define "number" more clearly. In other words, we need to identify the "criterion" (*Kennzeichen*) that decides in all cases whether a number is the same as another (GA 94 [§ 62]). This criterion, which Dummett calls a "criterion of identity",[42] will be the means we use to "recognize" a definite number "as the same again" (*als dieselbe wiederzuerkennen*), and so should provide the key to understanding what makes numbers the distinctive objects they are.

The sentence we need to consider, therefore, is not just "the number a (belonging to the concept F) is the same as the number b (belonging to the concept G)", but rather "the number a is the same as the number b *when* . . . ". If, however, this sentence is to be informative, and not just circular, then its "when" clause cannot itself make use of the phrase "the number a" or "the number that belongs to the concept F". The sentence must explain what it means for one number to be the same as another in terms that omit the word "number" altogether. By doing so, it will disclose the distinctive circumstance under which numbers, as opposed to other objects, are the same and so will guide us to the definition of number that we seek. The sentence will set out a criterion of identity for numbers, as we usually conceive them, but it will enable us to reach a better, indeed the proper, understanding of the term "number", whose use, as Dummett puts it, "has not yet been completely fixed".[43]

Frege finds the sentence he is looking for in book 1 of Hume's *A Treatise of Human Nature* (1739-40). Hume writes: "when two numbers are so combin'd, as that the one has always an unite answering to every unite of the other, we pronounce them equal".[44] Note that, according to Hume, in order to tell that two numbers are equal, we do not need to know what numbers – or rather what number – they are: we just need to match each *unit* in one number with a unit in the other (or each *thing* in one numbered collection with a thing in another). In § 70 of the *Foundations* Frege gives a concrete example to show what Hume has in mind:

> If a waiter wants to be sure that he is laying just as many [*ebenso viele*] knives as plates on the table, he does not need to count either the latter or the former, as long as he places a knife to the right of each plate, so that each knife on the table lies to the right of a plate.
>
> —GA 102

In this case, Frege continues, the plates and knives are "correlated with one another one-to-one" (*eindeutig einander zugeordnet*): each knife is correlated with a plate, and *vice versa*.[45] It is important to stress that any one knife is correlated with just *one* plate: the matching of knives and plates is *one-to-one*. According to Hume's principle, therefore, the *number* of knives is the same as the *number* of plates when the two sets of things are correlated with one another exactly – one-to-one. This is Hume's "criterion of identity" for numbers.[46]

Indeed, it is not just Hume's criterion, for Frege notes that the idea of defining the equality of numbers in terms of one-to-one correlation "seems recently to have gained widespread acceptance amongst mathematicians" (GA 95 [§ 63]). This idea thus provides a now familiar conception of the *equality* of, or between, numbers, on the basis of which we can reach a definition of number itself. Yet Hume's criterion needs to be adjusted somewhat to suit Frege's

purposes: for, as we know, Frege does not think that numbers apply directly to *things* (or that they apply to, or contain, mere units), but he believes that numbers should be ascribed only to *concepts*. Suitably adjusted, Hume's principle would thus state that the number belonging to the concept "knife" is equal to that belonging to the concept "plate", when the things falling under each concept are correlated with one another one-to-one.

Frege accepts that this principle is true, though he does raise a "concern" about it. One might argue, he notes, that "the relationship of equality does not hold only amongst numbers" and that, consequently, "it ought not to be defined specially for this case". On the contrary, one might think, there is a *general* concept of equality, and if we put that together with the concept of "number", we will know when numbers are equal; there is then no need for a special "criterion of identity" for numbers. Frege points out, however, that "for us the concept of number has not yet been fixed", so we cannot simply combine it with a general concept of equality in order to understand how numbers are equal. Our aim is rather to discover *what* a number is, and so to define the word "number", *by* first considering the distinctive way in which numbers are equal. As Frege puts it, our aim is, "by means of the concept of equality, taken as already known" – the equality of *numbers* – "to obtain that which is to be regarded as being equal" (GA 95 [§ 63]).

Frege's concern about Hume's principle is thus that we should perhaps not be employing such a principle at all. He allays this concern, however, by insisting that we need it after all: for our task is not to start from a settled conception of number in order to discover how numbers are equal, but to discover what a number is *from* the distinctive way in which one is equal to another. Number is thus the unknown element, and the equality between numbers the known one, not the other way around. We need a generally accepted principle or "criterion" of identity for numbers, therefore, if we are to define "number" itself. Of course, we must have a working idea of number already in order to identify how numbers in particular are equal; we can have such an idea, however, without being able to say exactly "what number is", as Frege thinks is proven by the example of mathematicians. This working idea of number thus points us to the criterion of identity for numbers and from the latter we can then derive a more precise definition of number.

This criterion of identity is not simply circular and question-begging, because it explains the equality between *numbers* attached to concepts in terms of the one-to-one correlation between *things*. Nonetheless, that criterion is itself presupposed by Frege without being derived and justified logically. He inherits it from Hume, and he endorses it because it unpacks the concept of equality between numbers that is "already known" (*schon bekannt*) and generally accepted by current mathematicians (GA 95 [§ 63]). He is unable, however, to provide any further justification for it.

To recapitulate: in setting out his account of number, Frege first presupposes the distinction between concepts and objects and ascribes numbers to the former but not the latter. He then seeks the definition of number by presupposing the context principle. This principle directs him to sentences in which number words are used and, more specifically, one number is equated with another. If such a sentence is, however, to help us understand what a number is, it must say *when* numbers are equal without using the word "number" in so doing; that is, it must set out a criterion of identity for numbers. This criterion will thus enable us to comprehend the distinctive kind of things that numbers are. The particular criterion that Frege takes from Hume, however, is never properly justified by him; he just assumes that it is appropriate because it spells out what the equality between numbers is generally taken to be. As we have seen, Frege places great stress on the importance of proof, and he aims to prove the laws of arithmetic by deriving them from laws of logic and the definitions of number and of the numbers. It appears, however, that the process of defining number itself need not be so rigorous: for it rests on principles that are themselves never proven but remain mere assumptions.

Frege is untroubled by this, though he is aware that his strategy for defining number might otherwise give cause for concern, as we have just seen. He allays that concern in the manner we have indicated, and in § 64 he provides examples to show that his strategy is by no means unprecedented. In geometry, Frege notes, "parallel lines are frequently defined as lines whose directions are equal"; and when we define them in this way, we assume that we know what the "direction" (*Richtung*) of a line is and that we can derive, from our understanding of the latter, an understanding of what it is to be "parallel" (GA 96). Yet geometry, for Frege (as for Kant), is rooted in a priori intuition and we have no direct intuition of a "direction". We can intuit a *line*, Frege insists; "but is the direction of a line distinguished in intuition from the line itself? Hardly!". We have no firm grasp of the "direction" of a line, therefore, on the basis of which to comprehend being "parallel".

We are able, however, to intuit one line's being parallel to another – "one does have an image [*Vorstellung*] of parallel lines" – and we can draw on this intuitive grasp of "being parallel" to understand what is meant by the "direction" of a line. In this case, the "direction" of a line is derived from its being parallel, rather than the other way around. The two are connected because lines point in the *same direction* when they are *parallel*; indeed, saying that two lines point in the same direction is just another way of saying that they are parallel, and *vice versa*. This gives us a criterion of identity for directions, even though we have no clear "intuition" of a direction itself.

If we assume that a "direction" is a logical object, like a number, then it must be capable of being recognized as the same again, that is, of being identified with another direction. We must be able to say, therefore, that "the direction of

line *a* is equal to [*gleich*] the direction of line *b*" (GA 97 [§ 65]). Yet the precise meaning of this sentence remains unclear, since we lack a clear conception of "direction". We can clarify its meaning, however, because we possess a criterion of identity for directions: we can thus say that the direction of line a is equal to the direction of line b *when* line a is parallel to line b. The equality or sameness of direction is thereby explained by expressing it in a sentence (or clause) that makes no mention of direction itself. In the same way, Frege contends, we can say that the "shape" of one plane figure is the same that of another, when the two figures are geometrically similar or "congruent" (*ähnlich*) (GA 96 [§ 64]).

The analogy between these two geometrical cases and that of number is evident. In all three cases, we start with a logical object, or rather an object-word, in need of definition: "number", the "direction" of a line or the "shape" of a figure. We then formulate the sentence that object a is the *same* as object b, and we then express the content of that sentence again in a new one (that follows the word "when"). Two things distinguish the second sentence from the first. On the one hand, the second sentence makes no mention of the objects in the first one, but asserts something of different objects: "lines" instead of "directions", and "things" or "objects" instead of "numbers". On the other hand, the second sentence does not just say of its objects *that* they are the same as, or equal to, one another (as the first one say of its objects), but it specifies the characteristic manner in which they are equal: it says that they are "parallel to" or "congruent with" or "in one-to-one correlation with" one another. In this way, it illuminates the distinctive character of the objects in the first sentence in a way that the first sentence itself does not. To say simply that the number a is the same as the number b is not informative, if, as Frege insists, "the concept of number has not yet been fixed" (GA 95 [§ 63]). It is to acknowledge that a number is a reidentifiable object, but to do no more than that; such a statement does not, therefore, make it clear how a number differs from a direction or a shape. However, to say that numbers are the same when objects are correlated one-to-one, whereas directions are the same when lines are parallel, is to distinguish a number from a direction – or rather it is to distinguish the way in which numbers are equal from the way in which directions are equal. It thus marks a step towards defining "number" and "direction" as distinct object-words.

Frege's strategy for defining number, or at least for moving towards such a definition, is thus not wholly unprecedented, because, in his view, analogous strategies are employed in geometry. In all such cases, the nature of a logical *object* is clarified by specifying, in a way that makes no mention of the object itself, what it means for that object to be *equal* to another. As Dummett writes, therefore, "the concept of parallelism is prior to that of a direction, so that the latter must be defined in terms of the former, and not conversely; by analogy, the concept of a number must be defined in terms of cardinal equivalence" (or one-to-one correlation), and not the other way around.[47] Yet the analogy between

Frege's geometrical examples and the case of number is not perfect. The direction of *line* a is the same as the direction of *line* b, when *line* a itself is parallel to *line* b; similarly, the shape of two figures is the same when the figures themselves are congruent. The case of number, however, is different: for it is not the case that the number of *concept* F is the same as the number of *concept* G when the two *concepts* stand in one-to-one correlation to one another. The concepts have the same number when the *objects* falling under them, such as knives and plates, are correlated with one another one-to-one. As we shall see in a moment, however, Frege formulates new sentences concerning direction, shape and number that all have the same logical form – a form that includes the distinction, introduced for the first time in his work, between a concept and the "extension" (*Umfang*) of that concept (GA 100 [§ 68]). To see what motivates this move by Frege, we need to consider a further concern he has about the strategy for defining number that he sets out in §§ 62-3. He raises this concern in connection with the concept of "direction", but it applies to any logical object, including number.

The concern is the following one. The claim that one line is *parallel* to another line explains what it means for the "direction" of the one to be the same as the "direction" of the other. This in turn sheds light on the distinctive nature of a "direction", for we see that sameness of direction is different from sameness of shape and sameness of number: each logical object is the same as another of its own kind in its own distinctive way. Parallelism, however, does not provide a criterion of identity or sameness "for all cases" in which a direction is involved; it provides such a criterion only for cases involving two objects, *each* of which is a *direction*. It tells us that direction a is the same as direction b, but it does not tell us whether direction a is the same as that which is *not* itself a direction. As Frege puts it, the criterion of parallelism cannot tell us "whether the proposition 'the direction of *a* is equal to *q*' is to be affirmed or denied, unless *q* itself is given in the form 'the direction of *b*'" (GA 99 [§ 66]).

"Being parallel" is equivalent, logically, to "having the same *direction*"; the fact that two lines are *parallel* does not, therefore, shed any light on what is *not* itself a direction. It does not tell us, for example, whether a given *number* is equal to direction a, or "whether England is the same as the direction of the Earth's axis" (GA 98 [§ 66]). Frege acknowledges that no one would ever confuse England with the direction of the Earth's axis; but, he insists, "that is no thanks to our explanation" for the latter cannot help us decide either way. This is not to deny that England, as a matter of fact, could be parallel to the axis of the Earth. Yet if the two were parallel, the *direction* of the one would be the same as the *direction* of the other; England itself would not be the same as the direction of the Earth's axis. Parallelism would not, therefore, establish an identity between a direction and something other than a direction.

The same applies to one-to-one correlation in the case of numbers. Such correlation between objects tells us that this *number* (belonging to concept F) is

the same as that *number* (belonging to concept G); but it does not, and cannot, tell us whether this or that number is the same as that which is not explicitly a number itself: for example, *Julius Caesar*. This is not to deny that Julius Caesar can be "correlated one-to-one" with another object, like the sofa in my living room (since there is only one of each). In this case, however, the *number* attaching to the concept "Julius Caesar" would be the same as the *number* belonging to the concept "sofa in my living room"; Julius Caesar himself would not be the same as the number of sofas in my living room. One-to-one correlation between objects would not, therefore, establish an identity between a number and something other than a number.

A criterion of identity thus enables us to judge that one object is the same as another object of the *same kind*: that two directions or two numbers are the same. It does not, however, enable us to decide whether an object is the same as, or different from, an object of a different kind: whether this direction is the same as, or different from, that number. The relation between a *direction* and a *number* cannot be directly converted into a relation between parallel lines or into the one-to-one correlation between objects; neither parallelism nor one-to-one correlation, therefore, can tell us whether the direction and the number are the same thing or different. Yet if these criteria of identity can neither identify this direction with that number, nor distinguish them from one another, *then they cannot tell us what either object itself is*: they cannot deliver a definition of either object, or indeed of any object at all. We still do not know, therefore, exactly what a number is.

This, however, is a serious concern: for Frege introduced the idea of a "criterion of identity" for numbers precisely in order to discover what a number is and so, finally, to define the word "number". His aim, as he puts it, was "by means of the concept of equality, taken as already known, to obtain that which is to be regarded as being equal" (GA 95 [§ 63]). The identity criterion for numbers, like all such criteria, has proven, however, to be incapable of delivering what Frege expected from it. It tells us what it means for one number to be equal to another number, but it does not tell us what any number, or number as such, *is*. This is because it does not tell us what it means for any number to be equal to, or more importantly to *differ* from, that which is not itself a number.

Frege's direct definition of number

Frege was able to allay his initial concern about the strategy he sets out in §§ 62-3 for defining number (2: 116). He is unable, however, to allay the further concern we have just considered. Accordingly, he decides (in § 68) to adopt a new strategy: to define number directly. The direct definition of number will not be a sentence of the form "the number of concept F is *equal* to the number of concept G, *when* . . . ", but it will say simply "the number of the

concept F *is the* such and such". This does not mean, however, that Frege now rejects his previous strategy and the idea of a "criterion of identity" altogether; on the contrary, to use Dummett's words, Frege "considers it to have been an essential aid to arriving at his conclusions".[48] (Dummett is talking more specifically about Frege's context principle, but, as I understand it, he also intends his words to refer to the idea of a criterion of identity.) The continuing importance of criteria of identity in Frege's project is clearly evident in § 68: for his new strategy for defining number draws directly on the criterion for the identity of numbers that he earlier found in Hume.

In § 68 Frege leads up to his direct definition of number by first considering how to define the term "direction". He has assumed since § 64 that there is a close analogy between "direction", "shape" and "number", so if he can explain how to define "direction" he will have found the key to defining "number". Moreover, he noted the failure of criteria of identity to yield definitions of objects while he was considering "direction" in particular, so it makes sense now to examine first how we might better define that term.

Recall that (according to §§ 64-5) when line a and line b are parallel to one another, the *directions* of the two lines are the same. In § 68 Frege now contends that other things are the same, too, namely the "extensions" (*Umfänge*) of certain concepts (GA 100).

At the start of § 68 Frege tacitly assumes that a statement asserting two lines – that is, two *objects* – to be parallel can be recast as a statement subsuming one object under a *concept*. He thus invokes once again the distinction between objects and concepts that he takes for granted earlier in the *Foundations*, and he converts the relation between objects into the subsuming of an object under a concept.[49] To say that line a is parallel to line b, therefore, is to say that line a falls under the concept "line parallel to line b". There are, however, numerous lines, besides a, that are parallel to line b, all of which thus fall under the concept "line parallel to line b". In § 68 the set of objects falling under a concept is called the "extension" of that concept (and after 1891 these objects are paired with truth-values in the extension of which they are members). The set of all the lines falling under the concept "line parallel to line b" thus constitutes the extension of that concept. Our sentence stating that line a is parallel is parallel to line b can, however, be reversed to state that line b is parallel to line a. In this case, b can be said to fall under, and belong to the extension of, the concept "line parallel to line a".

Now if lines a and b are, indeed, parallel to one another, then every line parallel to a will also be parallel to b and *vice versa*. The set of lines parallel to each will thus coincide. The only difference between the two sets will be that, strictly speaking, line a itself does not belong to the set parallel *to* line a, and line b does not belong to the set parallel *to* line b; otherwise, the sets are the same. These sets are, however, the *extensions* of their corresponding concepts;

these extensions must, therefore, also be the same. This brings us to Frege's first major claim in § 68:

> If line *a* is parallel to line *b*, then the extension of the concept "line parallel to line *a*" is equal to the extension of the concept "line parallel to line *b*"; and conversely, if the extensions of these two concepts are equal, then *a* is parallel to *b*.
>
> —GA 100

Frege now proceeds directly to his definition of "direction", though there appears to be a stage missing from his argument. The missing stage, I would suggest, is this: when line a is parallel to line b, the *direction* of a is the same as the *direction* of b; it is also the case that the *extension* of "line parallel to a" is the same as the *extension* of "line parallel to b"; it is logical, therefore, to propose that the direction of a is itself the same as the extension of "line parallel to a". This in turn gives Frege the definition of "direction" that he has been seeking, and it also indicates how we should define the analogous geometrical term "shape". "Let us therefore suggest", Frege writes, "the definitions: the direction of line *a* is the extension of the concept 'parallel to line *a*'; the shape of triangle *d* is the extension of the concept 'similar to triangle *d*'".

Two potential problems with these definitions immediately strike the reader. First, whereas the direction of line a belongs to *a* itself, the extension with which it is equated in Frege's definition strictly speaking excludes a (as we noted above), for it encompasses all the lines that are parallel *to* a. This need not make it impossible for the direction and the extension to coincide, but it demands clarification that Frege does not provide.

Second, the direction of line a is a single object and the extension of a concept is also a single object, as we can tell from the definite article; yet they are clearly different kinds of object, since an extension has other objects as its members, whereas a direction has no "members". As Weiner puts it, "an extension is supposed to be an object associated with a concept that has as its members precisely those things of which the concept holds".[50] An extension is thus structurally ambiguous, just as a "class" will be in Frege's later thought: it is a set or "aggregate" (*Menge*) of objects that is also an object in its own right (which is why there can be extensions with no members).[51] Frege's identification of the "direction of line a" with the extension of a concept is thus itself ambiguous. On the one hand, insofar as an extension is a set of objects, Frege equates the *single* direction of line a with a *multiplicity* of lines, each of which is parallel to line a. This equation of one object with many does not itself invalidate Frege's definition, but once again it demands clarification. On the other hand, insofar as an extension is a single object, Frege appropriately equates the single direction of line a with another single object. Yet this equation

appears more problematic when one bears in mind that an extension has *members*: for a direction is not the kind of thing that can have members. Again, Frege's definition calls for clarification that it does not receive.

Two responses to these last difficulties come to mind. The first is to suggest that Frege really wants to equate the direction of line a with what is *common* to all the lines in the extension of the concept "parallel to line a". This would make good sense from a non-Fregean point of view, since it is easy to see how all such lines "share" the same direction. Yet Frege clearly equates the direction with the extension itself and thus with the set of all lines parallel to line a, *not* with something common to them. Moreover, the direction of line a cannot be equated with what is "common" to a set of lines, since that would make it a property of the latter and the definite article indicates that it is an object, not a property. So although it may make sense to think of a direction as something shared by parallel lines, neither Frege's definition of "direction", nor his distinction between objects and properties, allows him to do so.

The second response is to remind ourselves that Fregean definitions are not meant to be statements of identity between objects that are both well understood, but that such definitions are intended to stipulate what is to be understood by the term being defined.[52] We should not, therefore, judge these definitions to be problematic on the basis of what we already understand a direction to be, but we should allow the concept of extension to give new meaning to the term "direction". Yet we should also remember that Frege's definitions are meant to be "useful for science" and thus to clarify terms with which scientists are already familiar (GA 90 [§ 57]), and this requirement is hard to satisfy if a "direction", as defined by Frege, ends up being significantly different from a direction as we normally conceive it. By this criterion a definition of "direction" that equates it with an object that has multiple "members" remains problematic.

Be all this as it may, Frege defines "direction" (and "shape") in terms of the extension of a concept, and then uses this model to propose a definition of number, which is what we are seeking. This definition has the advantage, from Frege's perspective, of being independent of the traditional idea that a number is a set of featureless units: it is directly derived from his distinction between concepts and objects (and thus between concepts and extensions), from his context principle and from his Humean criterion of identity for numbers. For Frege, such a derivation clearly adds strength to the definition at which he arrives. To Hegelian eyes, however, it means that the definition rests on *assumptions* that are never properly justified by Frege.

In Frege's view, numbers belong to concepts, and two numbers are the same when the objects falling under one concept are correlated one-to-one with the objects falling under another. One-to-one correlation in the case of numbers thus matches parallelism in the case of directions: in each case the former provides the criterion of identity for the latter. Now, as we have seen, to

say that line a is *parallel* to line b is to say that line a falls under the *concept* "line parallel to line b" and that it is thereby included in the extension of that concept. Accordingly, the sentence "the objects under concept F are *correlated one-to-one* with the objects under concept G" can be recast as "each object under concept F falls under the *concept* 'correlated one-to-one with an object under concept G'". This in turn means that the objects under F belong to the *extension* of the concept "correlated one-to-one with an object under concept G", along with all the other objects, under other concepts, that are also correlated with those under concept G. Conversely, the objects under G belong to the extension of the concept "correlated one-to-one with an object under concept F".

This now gives us the key to a new definition of number. In what I took to be the missing stage in Frege's argument for his definition of "direction", the sequence of thoughts was as follows. When line a and line b are parallel to one another, the *direction* of a is the same as the *direction* of b. Similarly, the *extension* of "parallel to a" is the same as the *extension* of "parallel to b", since all the lines parallel to the one are parallel to the other (if we ignore the fact that, strictly speaking, a and b cannot be parallel to themselves). It thus makes sense to propose that the direction of a is itself the same as the extension of "line parallel to a" (again, if we ignore the problems raised above [2: 122]). One can now imagine an analogous sequence of thoughts about number. So, when the objects under concept F can be correlated one-to-one with those under concept G, the *number* belonging to concept F is the same as the *number* belonging to concept G. Similarly, the *extension* of "correlated one-to-one with an object under F" is the same as the *extension* of "correlated one-to-one with an object under G", since all the objects correlated with those under F can also be correlated with those under G, and *vice versa* (if we ignore the fact that those under F or under G cannot be correlated with themselves).[53] It thus seems logical to propose that the *number* belonging to F is itself the same as the *extension* of "correlated one-to-one with an object under F". So our definition of number will be this:

> The number that belongs to the concept F is the extension of the concept "correlated one-to-one with an object under the concept F".

In § 68 Frege himself takes a more direct route to the definition of number. If we start with the definitions of "direction" and "shape", we have simply "to substitute for lines [*Geraden*] or triangles concepts, and for parallelism or similarity the possibility of correlating one-to-one the objects that fall under the one concept with those that fall under the other" (GA 100).[54] So we first take Frege's definition of direction: "the direction of *line a* is the extension of the concept '*parallel* to line a'". We then replace "line a" with "the concept F", and of course "direction of" with "number that belongs to"; and we replace

"parallel to" with "correlated one-to-one with an object under". This then yields the definition of number given above (though Frege does not provide that definition explicitly himself).

Compared to the definitions of direction and shape, however, the definition of number is a bit of a mouthful. Frege thus proposes a shorter way of expressing the thought of the one-to-one correlation of objects. When the *objects* under concept F can be correlated one-to-one with the *objects* under concept G – like the plates and knives in Frege's example (see GA 102 [§ 70]) – then, Frege declares, we can say "for short" (*der Kürze wegen*) that *concept* F is itself "*equinumerous*" (*gleichzahlig*) to *concept* G. Conversely, concept G is equinumerous to concept F, when the objects under the former can be correlated one-to-one with those under the latter. This verbal shorthand yields Frege's "official" definition of number:

> The number that belongs to the concept F is the extension of the concept "equinumerous to the concept F".
>
> —GA 100 [§ 68]

It is important to emphasize, however, that this formulation of the definition of number is merely a shorthand version of the definition I gave above and has the same meaning as the latter. Frege's official definition is thus not be taken at face value but is to be read as *standing for* the first definition. This is because the phrase "equinumerous to the concept F" is itself not to be taken at face value. Frege is quite explicit about this. We call one concept "equinumerous" to another, when there is the possibility of correlating the objects under one with the objects under the other; but "this word" – "equinumerous" – should "be regarded as an arbitrarily chosen form of expression, whose meaning is to be gleaned not from its linguistic composition but from this stipulation" (GA 100 [§ 68]). To say that concepts are "equi-numerous" – *gleich-zahlig* – is thus not simply to say that the *same number* – *die gleiche Zahl* – belongs to both, even though that is what the composition of the word suggests. It is to say what Frege has declared it should say: that the objects under the one concept can be correlated one-to-one with the objects under the other concept. This means that when we say two concepts are equinumerous, we are not actually talking about those *concepts*, but we are talking about the *objects* that fall under them. This point is confirmed in § 73 of the *Foundations*. Frege refers there to the proposition "if the concept H is equinumerous to the concept F, it is also equinumerous to the concept G", and he states explicitly that this proposition "amounts to saying [*kommt darauf hinaus*] that there is a relation which correlates the objects falling under the concept H one-to-one with the objects falling under the concept G, and vice versa" (GA 106). We must bear this clearly in mind as we consider the meaning of the term "extension" in Frege's official definition of number.

If we take that definition at face value, then the extension referred to in it is, unambiguously, the extension of the concept "equinumerous to the concept F". This concept, however, is a second-level concept under which other first-level concepts – G, H and so on – can fall. Those first-level concepts thus constitute the extension of the second-level concept.

As Frege notes in § 53 of the *Foundations*, there is a difference for him between subordinating a first-level concept to another first-level concept and subsuming a first-level concept under a second-level one (GA 87). If we say that a horse is a mammal, then we subordinate the concept "horse" (or "being a horse") to that of "mammal": we say that "being a horse" entails "being a mammal". This statement can then be converted into one that subsumes an *object* under both concepts, namely "if x is a horse, x is a mammal"; but it does not subsume one *concept* under the other, for it does not say that the *concept* "horse" is itself a mammal. We can, however, subsume the first-level concept "horse" under the second-level concept "being easily attained", and in this case we say something about that first-level concept as a *concept*, not something about being a horse (though in so doing we actually turn "the concept *horse*" into an object and so turn the second-level concept into a first-level one under which that "object" is subsumed) (see FR 184 / ÜBG 50).[55]

When Frege's official definition of number is taken at face value, the concept "equinumerous to the concept F" is clearly a second-level concept that subsumes under it the *concepts* G, H and so on, not Gs and Hs themselves. In this case, when the plates on Frege's table are correlated one-to-one with the knives, we say of the *concept* "plate" that it is "equinumerous to the concept 'knife'", but we say nothing directly about the plates (or knives) as such. The items that fall under the concept "equinumerous to the concept 'knife'", or "equinumerous to the concept F", are thus (first-level) concepts (such as "plate"), not objects (such as plates). This in turn means that the *extension* of the second-level concept has first-level concepts, not objects, as its members.

So, to take Frege's official definition at face value, and to define a "number" straightforwardly as the extension of the second-level concept "equinumerous to the concept F", is to identify the number with the set (or class) of *first-level concepts* falling under that second-level concept. This, as far as I can see, is what Dummett does in his study of Frege's philosophy of mathematics. Dummett writes that "the extension of a second-level concept is a class of concepts, just as the extension of a first-level concept is a class of objects"; and he goes on to claim that "the *Grundlagen* notion of a class", or extension, is that of one "whose members are concepts".[56] He makes the same point when he contrasts the conceptions of number in the *Foundations* and the *Basic Laws*: despite the closeness between the two conceptions, in *Basic Laws*, he maintains, "number is given as a class of classes *rather than of concepts*".[57] In the later text, therefore, a number is conceived as a set of extensions, or set of

sets of objects, whereas in the earlier *Foundations* it is understood as a set of concepts.

In my view, however, Dummett doesn't see that Frege's official definition of number in the *Foundations* is not to be taken at face value. As noted above, saying that one concept is "equinumerous" to another, for Frege, is simply a shorthand way of saying that the *objects* under the first concept are, or can be, correlated one-to-one with the *objects* under the second concept; it is thus not actually to say something about the *concepts* themselves. Accordingly, in Frege's definition of number, being "equinumerous to the concept F" is not to be taken as a concept that subsumes other concepts, and, *pace* Dummett, such concepts are not to be taken to constitute its extension. Frege defines the number belonging to concept F officially as the extension of the concept "equinumerous to the concept F", that is, as the extension of a second-level concept. Yet he doesn't mean by this that the number is the set (or class) of all *concepts* that are equinumerous to concept F. What he means – what his definition stands for – is what is stated in the first version of the definition given above: that the number belonging to the concept F is the extension of the concept "correlated one-to-one with an object under the concept F". This extension comprises objects, rather than concepts. A number, for Frege in the *Foundations*, is thus itself a set of objects.

Note, however, that the number belonging to concept F is not simply the set of objects falling under *that* concept, but the set of objects that are, or can be, *correlated* one-to-one with those falling under it. These objects fall, of course, under other concepts, and so are grouped into sets that are the extensions of those other concepts. They include the set of objects under concept G that are correlated with those under concept F, the set under concept H that are so correlated, and so on. A number is thus the set of all those sets of objects that are correlated with those under concept F. Number, therefore, is conceived in the *Foundations* in the same way Dummett takes it to be conceived in the *Basic Laws*, namely as "a class of classes" or as an extension whose members are extensions (whose members in turn are objects).[58] This conception of number cannot be reconciled with the letter of Frege's official definition. It is the conception that emerges, however, if we take that official definition to be a shorthand version of the proper definition, and if we model the latter directly on the definitions Frege gives of "direction" and "shape".

Frege's definition stipulates how the word "number" is now to be understood. This definition, however, is meant to be "useful in science" and so should overlap in some way with our working conception of number. There is one significant respect, however, in which number, as Frege defines it (on my reading), appears to be at odds with that working conception. The problem is that a Fregean number appears to encompass many more objects than a usual number does. A number as it is usually conceived contains a determinate amount of objects (whether or not we conceive them to be empty units): 1

contains one more than 0, 2 one more than 1, and so on. A Fregean number, by contrast, appears to encompass a large range, indeed possibly an infinity, of objects. This is because it is identified with the set of *all* the sets of objects that are, or can be, correlated one-to-one with the objects under its corresponding concept. There are, however, two ways in which the apparently excessive internal richness of a Fregean number can be reduced.

The first is to equate number with what is *common* to each set of correlated objects. The objects under concept G are correlated one-to-one with those under concept F, and so are those under concepts H, I, J and K. Yet instead of thinking of the number belonging to concept F as the extension that includes all these sets of correlated objects, we could equate it with the *property* that such sets share: each set of objects is correlated with those under concept F in the same way and that "way" would be the number common to all. As is the case with direction, however, number cannot be understood like this by Frege, since *the* number of concept F is clearly an object, not a property. It has thus to be understood as the extension of a concept, not as what is common to the objects in that extension.

The second way of reducing the internal richness of a Fregean number seems to me to be permitted by Frege's logical commitments, but it is unclear whether Frege himself has it in mind. What follows, therefore, is merely a suggestion as to how a Fregean number could be understood – a suggestion that may well be mistaken.

The extension, with which Frege equates the number of the concept F, does not just contain the objects under concept F, or those under concepts G and H taken by themselves. It contains the objects under G and H insofar as they are, or can be, *correlated* with those under F; that is to say, it contains *pairings* of objects.[59] These pairings, however, multiply the more concepts we bring to the situation: so we have one set of pairings if we just have the objects under concepts F and G, but two sets if we add the objects under concept H, and so on. Since there may be an infinity of concepts whose objects can be correlated one-to-one with those under F, the extension with which a number is identified potentially contains an infinity of pairings of objects. It is not necessary, however, to conceive of the extension in quite this way: for we can think of an object under concept F as correlated with objects under concepts G, H, I, J, K *all at the same time*. In this case, there is in fact just *one* correlation of the object under concept F with objects under the other concepts. An object under concept F is thus not correlated first with one under concept G and *then* with one under concept H, but it is correlated just *once* with an object under concept G *and* one under concept H, and so on. This remains the case, however many other concepts and sets of objects one adds: for every object under concept F there is always just *one* one-to-one – or one-to-one-to-one, or one-to-one-to-one-to-one – correlation.

The set of objects that fall under, and belong to the extension of, the concept "correlated one-to-one with an object under concept F" can thus be understood in two ways. On the one hand, it can be conceived, straightforwardly, as the set of all the sets of objects under other concepts that are, or can be, correlated with those under F. We can represent this set of sets like this:

Set F – G	+	*Set F – H*	+	*Set F – I*
plate (F) – knife (G)		plate (F) – spoon (H)		plate (F) – fork (I)
plate (F) – knife (G)		plate (F) – spoon (H)		plate (F) – fork (I)
plate (F) – knife (G)		plate (F) – spoon (H)		plate (F) – fork (I)
.		.		.
.		.		.
.		.		.

In this case, number – which Frege identifies with the extension we are considering – will itself be a set of different sets of correlated objects, or in Dummett's description (which he wishes to reserve for the *Basic Laws*) "a class of classes".[60] This, however, leaves us with the problem noted above: namely, that a Fregean number appears to encompass many more objects than a usual number does.

On the other hand, the same sets of objects under other concepts can be understood as correlated with those under concept F all at the same time, and in this case they form together a *single* set of correlated objects. We can represent this set like this:

Set F – G – H – I – J . . .
plate (F) – knife (G) – spoon (H) – fork (I) – glass (J) . . .
plate (F) – knife (G) – spoon (H) – fork (I) – glass (J) . . .
plate (F) – knife (G) – spoon (H) – fork (I) – glass (J) . . .
.
.
.

When number is identified with this single set of *correlated* objects, it still differs from number as usually conceived, since we usually think of a number as a set of objects (or units) *tout court*. Yet it differs a lot less than when it is conceived as a set of many sets of such objects: for the number of concept F, which is *one* thing, is now clearly equated with the *one* set formed by the objects correlated with those under F.

These two ways of understanding number are not in conflict with one another, but the second merely refines the first in order to bring it more in line with our usual working conception of number. This refinement is, I think,

permitted by Frege's logical commitments, but whether he might have it in mind is not at all clear.

What is important, however, is that, according to either understanding, a number, for Frege, is a set of objects, not concepts. His official definition of number makes it look as though a number is a set of concepts, but this impression is misleading, as I argued above. This is not to suggest that Frege should, or would if pressed, repudiate his official definition of number. In Frege's view (as I understand it), it is quite in order to say that "the number that belongs to the concept F is the extension of the concept 'equinumerous to the concept F'" (GA 100 [§ 68]), *provided* that one remembers two things. First, to call one *concept* "equinumerous" to another is simply a shorthand way of saying that the *objects* under the one can be correlated one-to-one with the objects under the other. Frege makes this clear in §§ 68 and 73, as we have seen, and he also emphasizes the point in § 72:

> The expression "the concept F is equinumerous to the concept G" *is to mean the same* [*sei gleichbedeutend*] as the expression "there is a relation φ, which correlates the objects falling under the concept F one-to-one [*eindeutig*] with the objects falling under G, and *vice versa* [*beiderseits*]".
>
> —GA 105 [§ 72], emphasis added

Second, it is these correlated objects – not concepts – that constitute the "extension" of the concept "equinumerous to the concept F". For Frege, therefore, the *number* belonging to a concept F is the *set of correlated objects* (or the set of sets of such objects) that is associated with that concept.

It is worth emphasizing at this point the most obvious peculiarity of Frege's definitions of both "number" and "direction": namely, that their objects belong to items that themselves come in pairs and so are in *relation* to one another. A direction, for Frege, belongs to a line that is *parallel* to other lines: for the direction of line a is simply the extension of the concept "parallel to line a". Similarly, a number belongs to a concept, under which fall objects that are *correlated* one-to-one with those under another concept: for the number that belongs to the concept F is simply the extension of the concept "correlated one-to-one with an object under the concept F". It would appear, therefore, that if there were only one line, we could not say what it would be for that line to have a "direction", and if there were only one concept we could not say what it would be for that concept to have a "number"; indeed, it is not clear that there would be such a thing as a direction or number at all. (This is not to say that if all we see are books, we can't count them; but by assigning them, or their concept, a number, we necessarily assume that they can be matched up with a different set of things.)

This feature of Frege's definitions stems from the fact that they are derived (in part) from the criteria of identity for their respective objects: for these criteria

enable us to determine when an object is the same as *another* object, that is, when it is the same one *again*. In Frege's view, for an object, such as a direction or number, to have an identity, it must be *re*-cognizable and *re*-identifiable; and, according to the context principle, the "re-cognition" (*Wiedererkennen*) of such an object must be expressed in a sentence that equates the object with *another* one (that is thereby declared to be the same object "in another guise") (GA 94, 98 [§§ 62, 66]).[61] Furthermore, while the object is still undefined, a criterion is required to explain in different terms what it means for that object to be the "same" as another: so, for example, the direction of line a is said to be the same as that of line b, when line a is *parallel to* line b. For Frege, therefore, an object we seek to define must be conceived in relation to another that is the same as it is; and, as noted above, it must be understood to belong to items – lines or concepts – that themselves come in *pairs* (or in threes or fours . . .). Frege's distinctive, indeed peculiar, definition of number is thus determined by the assumptions he makes about objects, the criteria of their identity and the context principle.[62]

These assumptions in turn help explain the significant differences between Frege's conception of number and Hegel's. Hegel derives his conception of number in the course of his presuppositionless, immanent logic. He does so by showing that quality makes necessary the empty, self-relating one, and that the one in turn gives rise to quantity and number. A number, for Hegel, as for the Greeks before him, is thus a set, or unified aggregate, of units that are both featureless and independent of one another (or "monadic"). Frege, by contrast, reaches his conception of number on the basis of his assumptions about objects and criteria of identity. Accordingly, a Fregean number comprises objects that are qualitatively distinct from one another (rather than featureless) and correlated with one another (rather than "monadic"). The differences between a Hegelian and a Fregean number are thus not just contingent, but are intimately connected to the different ways in which each philosopher derives the idea of number within (or from) his logic.

After providing his new and distinctive definition of number, Frege has one more thing to consider before turning to the individual numbers, 0 and 1. He has to show that his definition of number does not cease being true just because *no* object falls under the concept to which a number belongs. As we have seen, the concept F, for Frege, has a definite number insofar as every object that falls under it can be correlated one-to-one with an object under concept G, and *vice versa* (just as a line has a definite direction when it is a parallel to another line). Yet this immediately raises the question: can the concept F still have a number "when no object at all falls under *F*" (GA 104 [§ 71])? At first sight, the answer would appear to be no, since there appears to be no sense in talking about "every" object falling under a concept if *no* object falls under it.

This, however, is not how Frege understands the matter. In his view, even if no object falls under concept F, we can still say that *every* object under concept

F can be correlated one-to-one with an object under G. It's just that whatever object we choose, it will *not* fall under concept F; or, as Frege puts it, "'*a* falls under *F*' is always to be denied, whatever *a* may be" (GA 104 [§ 71]). (Similarly, when no object falls under concept G, we can still say that every object under concept G is correlated with one under concept F.) Even when no object falls under concept F, therefore, that concept can still have a number, since the condition for its having one has not been violated: for we do not now say that *no* object falling under concept F, or *not every* object under that concept, can be correlated with one under concept G.

Frege's definition of a number thus remains valid, even for a concept with no objects. It is important for Frege to have shown this, for otherwise it would be unclear how a concept could have the number 0, that is, how we could say (as we might want to) that there are zero "plates" – or no "plates" – on the table when every object on it falls under the concept "fork". In Frege's view, 0 is a number in precisely the same sense as 2 and 3. His definition of number would thus be of no value if it were not compatible with 0 being a number: for "what does not fit 0 or 1 cannot be essential for the concept of number" (GA 79 [§ 44]). In § 71, however, Frege does not define "0" itself. He simply shows how it is possible, logically, for there to be the number 0 at all, given his definition of number as such. He provides the definition of "0" in § 74.[63]

Frege's definitions of 0 and 1

For Frege, a number belongs to a concept F and is the *extension* of "equinumerous to the concept F". In this respect, every number is the same: it is a number (*Anzahl*) at all because it is such an extension. How then are we to distinguish the individual numbers (*Zahlen*) from one another? Obviously, 0, 1, 2 and 3 are different numbers, but how are we to distinguish and define each one *logically*? We do this, in Frege's view, by assigning each to a quite specific concept. This concept is not just one to which any number can belong (such as "plate" or "knife"), but it is *the* concept that gives the number its distinctive logical character. Each number can thus be defined as the number that belongs to its specific concept. In the case of 0, Frege maintains, the relevant concept is that of being "unequal to oneself" or "not identical with itself" (*sich selbst ungleich*). "0" is thus to be defined as "the number that belongs to the concept 'not identical with itself'" (GA 107 [§ 74]) (and it is a number at all because it is the *extension* of "equinumerous to the concept 'not identical with itself'").

Frege connects 0 to this concept because, as he puts it, *nothing* falls under it (and its extension is thus empty). Strictly speaking, Frege should avoid using the word "nothing" here, since it can be understood as just another word for zero; he should say instead that the number 0 belongs to the concept "not identical with itself" because, "whatever *a* may be, the proposition holds universally that

a does not fall under that concept" (GA 88 [§ 55]).[64] Yet, provided we understand that this is in fact what we mean when we say that "nothing" falls under the concept, there is no harm in using this expression, and Frege uses it here without further comment. He does note, however, that the concept "not identical with itself" is not the only one he could have chosen for the task at hand: "for the definition of 0 I could have taken any other concept under which nothing falls" (GA 108 [§ 74]). Yet he needs a concept that lacks an object, not just contingently, but as a matter of logical necessity, if his definition of "0" is to be grounded in logic alone. In his view, therefore, we must define the number 0 as the one that belongs specifically to the concept under which, logically, no object *can* fall because it contradicts the very concept of an object (which is precisely to be self-identical).[65]

From a Hegelian point of view, however, the concept of being "not identical with itself" cannot be used to define 0 in the way Frege hopes, for *every* object falls under that concept to some extent. This is not to deny that objects have their own identity and are "self-relating"; but, as I noted in chapter 3 of this volume, every object is also *non*-self-identical, since it owes its identity in part to what is *not* itself, that is, to other things or to a moment of internal negation and difference.[66] By defining 0 as the number that belongs to the concept "not identical with itself", Frege thus inadvertently turns it into the largest number conceivable, namely the number of all the things there are or ever could be – the number (or quantum) that, as we shall see, is for Hegel a logical impossibility.[67] To put the point another way, to define 0 as the number that belongs to the concept "not identical with itself" is no different from defining it as the number that belongs to the concept "identical with itself" (or to the concept "( ) is an object"). This is not to say that Hegel himself conceives of 0 in this way: on his conception of number, 0 is the number from which all component units have been removed and so is clearly the one with the least, as opposed to the greatest, content. To Hegelian eyes, however, *Frege's* definition of 0 turns it into the (impossible) number that encompasses everything (actual or possible).

Frege can, of course, respond by saying that his definition fits 0 perfectly, *if* we assume, as he thinks we must, that no object is "not identical with itself". Yet he never justifies this assumption, but simply treats the law of identity and the self-identity of objects as logically primitive.[68] The claim that his definition of 0 clarifies and defines anew what we normally understand 0 to be (rather than the number of all the objects there could be) thus itself remains without proper logical justification, because it rests on a logic and an ontology of objects that are simply assumed, dogmatically, to be true.

Frege's failure to define 0 in a way that keeps it distinct from the (impossible) largest number has a direct bearing on his conception of the number 1. "In order to reach the number 1", Frege states, "we must first show that there is something that immediately follows 0 in the series of natural numbers" (GA

110 [§77]); we can then call this new number the number "1". Yet, since "0", as defined by Frege, is the largest number, 1 will be the number following the largest number, which is incoherent. The number 1, as Frege conceives it, thus proves to be even more impossible (if one can put it that way) than his 0.

This, of course, is how Frege's conception of "1" looks to Hegelian eyes; but it is not how Frege sees the matter. He thinks he has defined 0 successfully and that the number following 0 can thus properly be called "1"; and he thinks he has defined 0 successfully, because he assumes – as all "understanding" does – that no object is "not identical with itself" (and he dismisses as "mad" anyone who disagrees) (FR 203 / GGA xvi).

How, then, does Frege define the number 1? He first sets out (in § 76) what it means logically for one number immediately to succeed another at all, and his argument recalls one he presented earlier in § 55 (see 2: 106-7). If there is a concept F under which an object x falls, and the number n belongs to this concept, and if the number m belongs to the concept "falling under *F* but not equal to *x*", then, Frege writes, "this is to mean the same as '*n* follows immediately upon *m* in the natural number series'" (GA 110 [§ 76). Frege's text at this point is concentrated, but the point he is making is relatively straightforward. Recall that if an object falls under the concept "plate" and the number 5 belongs to that concept, then we can say that there are *five plates*; similarly, when an object x falls under the concept F and the number n belongs to that concept, we can say that there are *n Fs*. Now consider the objects falling under the concept "falling under F but not equal to x" – a concept to which the number m belongs. These objects are all those that fall under F but that are not x, and there are *m* of them. With x, therefore, there are n Fs, and without x there are m Fs. According to Frege's stipulation, this is to mean that n follows immediately upon m in the natural number series (since they differ by just one object: x).

Frege now considers the number 0 in light of this stipulation (in § 77). He first takes the concept "identical with 0" and notes that 0 itself falls under it (since 0 = 0). This concept "identical with 0" corresponds to the concept F in § 76, and 0 – which, as a number, is itself an *object* – corresponds to the object x; the number belonging to this concept thus corresponds to the number *n*. Then Frege takes the concept "identical with 0 *but not* identical with 0" and notes that no object at all falls under this concept, since no object can be both identical with and not identical with 0. This, of course, includes 0 itself, since (in Frege's view) it certainly cannot not be identical with itself. The concept "identical with 0 *but not* identical with 0" clearly corresponds to the concept "falling under F *but not* x" in § 76, so the number that belongs to it must correspond to the number *m*. The number belonging to the concept "identical with 0" (which corresponds to n) must, therefore, follow immediately upon the number belonging to the concept "identical with 0 *but not* identical with 0"

(which corresponds to m). We know, however, what the value of this latter number (m) is: since no object – not even 0 – falls *under* the concept "identical with 0 *but not* identical with 0", the number belonging *to* this concept is 0. The number (n) of the concept "identical with 0", under which 0 itself falls, must therefore follow immediately upon 0 in the natural number series. Frege then stipulates that "1 is the number that belongs to the concept 'identical with 0'", and so concludes that "1 follows immediately upon 0 in the natural number series". Note that Frege could have stipulated that "16" or "37" be the name of the number belonging to the concept "identical with 0"; but his definition is meant to clarify a word we already use, so it makes sense to attach the number "1" to the concept under which, logically, just one object – namely, 0 – falls. (A similar point can be made about every number word, including "0": each one could be defined in all manner of ways, but it makes sense for its definition to clarify the way the word is normally understood. Accordingly, "0", rather than "16", is attached by Frege to the concept under which no object falls.)

Frege does not go on to provide definitions of the numbers beyond 1, but it is easy to see how one can do so. Since "1" is the number of the concept "identical with 0", "the next step", as Shapiro writes, "is to define the number two to be the number of the concept 'either identical to zero or identical to one', and so on for the rest of the natural numbers".[69] Frege could have taken this "next step" himself, but instead he provides a general argument to show that every number in the natural number series is succeeded by another and that, accordingly, "there is no last member of this series" (GA 114 [§ 82]). After this, he considers Cantor's idea of infinite or "transfinite" numbers, and highlights some differences between his own views and those of Kant (on analytic judgements and the nature of objects). He then brings the *Foundations* to a close by casting "a brief" (but helpful) "glance back over course of our investigation" (GA 116-22, 135-9 [§§ 84-9, 106-9]).

We will leave Frege's last thoughts to one side, however, since we have done enough to show that he derives and defines the natural numbers using the resources of pure logic, as he conceives it. He defines number (*Anzahl*) as such, and then the first two numbers (*Zahlen*), using logic alone. This then puts him in a position, on the basis of his definitions and of primitive logical laws, to derive mathematical principles, such as the law of associativity, and specific mathematical equations; and by doing this he can prove that arithmetical truths are analytic. Frege does not provide such proof itself in the *Foundations*; yet by defining "number" and the first two numbers logically, he shows it to be "probable" that arithmetic is analytic, and in so doing he lays down (what he takes to be) the logical foundations of arithmetic (GA 119, 138 [§§ 87, 109]).

Note, by the way, that, from a Hegelian point of view, Frege's definition of "1" is in one sense less problematic than his definition of "0". The latter is problematic because it assumes that *no* object is "unequal to itself", when in

fact every object is non-self-identical to some extent. Frege's definition of "1", however, assumes merely that the *number* 0 is identical with itself. This is less objectionable to Hegelian eyes, since numbers, for Hegel, are more radically independent, and in that sense more explicitly "self-identical", than other objects (if not unambiguously self-identical). In particular, they are not determined by anything other than their own amounts. As Hegel writes,

> Determinacy through number – how big something is – does not require being distinguished from something else of magnitude, as if to the determinacy of one thing belonged its magnitude and that of another, for the determinacy of magnitude as such is a limit determinate for itself, indifferent and simply related to itself; and in number this limit is posited as enclosed in the one that is for itself.
>
> —SL 182-3 / LS 231-2[70]

This is not to deny that numbers are in certain respects contradictory and thus not just simple "self-identical" entities.[71] Yet cardinal natural numbers, which is all that Hegel has derived so far, are self-contained in the manner just described in the quotation and *in that sense* "identical with themselves"; for Hegel, therefore, just as for Frege (who also has cardinal natural numbers in mind), 2 = 2, 1 = 1 and 0 = 0. The problem with Frege's definition of "1", from a Hegelian perspective, is thus not principally that it takes 0 to be identical with itself. It is that 0 has previously been defined in a way that turns it into the impossible "largest number", so 1, as the immediate successor of 0, becomes an equally (or even more) impossible number.

Yet, as I have insisted throughout this chapter and the last, Hegel would have a more fundamental objection to Frege's whole account of number in the *Foundations*: namely, that it rests on presuppositions that, by Frege's own admission, are never justified. These include the logical distinction between a concept and an object (and the related distinction between a concept and its extension), as well as the context principle. Since Frege's project is to derive arithmetic from logic, it is understandable that he would seek to provide purely logical definitions of number and the numbers. From Hegel's perspective, however, it is illegitimate to base logic, as Frege does, on unjustified presuppositions. The principal reasons why this is so will be familiar.

First, such presuppositions are at odds with the demand for *proof* (and justification) in philosophy and so must be avoided in the latter. Hegel agrees with Frege that "science demands that we prove whatever is susceptible of proof" and not simply take things for granted (FR 310 / LM 94).[72] He also agrees that the starting point for proof cannot itself be proven. For Hegel, however, this does not mean that we have to begin from an unjustified, unproven presupposition after all – a presupposition that should be proven but can't be.

In his view, as just noted, such presuppositions must be avoided in philosophy. The fact that the starting point cannot be proven can thus only mean one thing: that we must begin from that which presupposes nothing and so *requires no proof*. As we know, the only thought that meets this description, in Hegel's view, is one that is utterly indeterminate: the thought of pure being (which immediately converts itself into the thought of nothing).[73] The thought of pure being does not require proof, and so is the appropriate beginning of logic, because there is nothing determinate in it to be proven: it is completely empty and vacuous. By contrast, "what is lacking if we make something concrete the beginning is the proof [*Beweis*] which the combination of the determinations contained in it requires" (SL 55 / LS 68). By "concrete" Hegel means here any thought that is in some way determinate. Accordingly, this includes the thoughts that Frege regards as primitive (such as the law of identity and the thought that concepts and objects are distinct). The problem with Frege's logical assumptions, therefore, is not just that they are not proven, but that they lack the proof that their (albeit fairly minimal) determinate content requires them to have. In other words, they are unjustified assumptions *that should be justified* (or otherwise suspended).

The law of identity, presupposed by Frege, is actually proven by Hegel in the course of his logic – first, because he shows in the doctrine of being that every something is self-relating and so is *itself*, and second because he shows in the doctrine of essence that identity as such is necessary to both being and thought. The problem, however, from Frege's point of view, is that Hegel thereby also proves that identity is only half the story: for every something is also determined to be what it is by other things and so is not just itself, and identity is inextricably bound up with non-identity and, indeed, contradiction. (As we have just noted, cardinal numbers differ from other objects in being more explicitly self-identical; yet, as we will see later, they also *lose* some of their self-identity when they mutate logically into ordinal numbers and then become moments of ratios.)

The first Hegelian objection to Frege thus draws on the latter's own commitment to proof and is that Frege is simply not rigorous enough in banning from his logic any thought that should be, but is not, proven. The second objection arises from a commitment of Hegel's that Frege does not (or might not) share. This is the distinctive post-Kantian commitment to think *freely* and *self-critically* and so to take nothing determinate on mere authority or as simply given.[74] Yet this commitment also requires that logic start with indeterminate being (as, indeed, does the commitment to radical scepticism that Hegel inherits in part from the Greeks).[75] The Hegelian charge against Frege thus remains the same: he begins logic from determinate assumptions about identity, concepts and objects, when he should begin from pure being. In this case, however, the imperative to begin from pure being arises from the demand for freedom and radical self-criticism in thought, rather than from the demand for proof.[76]

Of course, the Hegelian charge against Frege carries real weight only if Hegel's thought is itself systematically presuppositionless in the manner I have described in this study. Otherwise, Hegel simply begins from a different set of unjustified presuppositions, and there is no reason to prefer his logic to Frege's: for, as Hegel himself states in the *Phenomenology*, "*one* bare assurance is worth just as much as another" (PS 49 / 60). Many critics of Hegel dismiss out of hand his claim to think without systematic presuppositions, and if they are right then my Hegelian critique of Frege loses much of its force (though it can still be true that philosophy *should* begin without such presuppositions).

Contrary to Hegel's critics, however, I take Hegel's claim seriously, and I have tried to defend that claim, against those critics, here and elsewhere. If, as I argue, Hegel's logic is indeed free of systematic presuppositions, in a way that Frege's clearly is not, and if Hegel succeeds, as I think he does, in deriving the concept of number immanently from his indeterminate starting point, then, necessarily, his account of number is more rigorously derived and so more justified than Frege's. This is not to deny that Frege's conception of number may prove, and has proven, "useful" in various ways; nor is it to deny the impressive consistency with which he derives that conception of number, and his definitions of the individual numbers 0 and 1, from his own assumptions. Nonetheless, since Frege's conception is grounded precisely on those assumptions – about logic, concepts and objects – it represents no serious philosophical advance over Hegel's radically presuppositionless conception of number and does not render the latter obsolete. (Nor, we should recall, does Frege's critique of the traditional Greek conception of number, which is close to Hegel's conception, meet its target: for, as I argued above, it, too, rests on assumptions about "objects", as well as on a failure to understand the logical character of the "one".)

In the last three chapters, I have tried to do more than merely assert that Frege's thought is question-begging. I have endeavoured to show in some detail exactly how his unquestioned assumptions about identity, concepts and objects underpin his whole account of number. It is now time, though, to leave Frege behind and to resume our study of Hegel's account of number.

CHAPTER SIX

Extensive and Intensive Magnitude

EXTENSIVE MAGNITUDE

Quantity, for Hegel, is the continuity formed by the self-externality of the one. It is the continuous unity of discrete ones. It does not just exist as such, however, but takes the form of limited, determinate units of itself, or quanta. A quantum is in turn a number (*Zahl*) when its determinacy resides specifically in the *amount* (*Anzahl*) of units it contains (SL 167-9 / LS 212-14). Numbers are thus differentiated by their respective amounts as 3, 4 or 5. Yet they are also *unities* formed by the units within them and, as such, are all the same. The unity of the number, however, stands in a twofold relation to the number's amount. The unity differs logically from the amount; yet the latter is not independent of, but belongs to, the number and so falls within its unity. Indeed, the number's amount actually coincides with its unity and thereby constitutes its "*being-determinate-in-itself*", or the determinacy the number has within itself quite apart from its relation to other numbers (SL 170 / LS 215).[1] Insofar as a number is a unity in this second sense – namely, with an internal amount – it is an extensive unity, or what Hegel calls an *extensive magnitude*, just like every other number (SL 182 / LS 231).

The logical differences between a quantum, a number and an extensive magnitude are subtle and easy to overlook. A quantum is simply a limited unit of quantity and, as such, all quanta are alike. A number, by contrast, is a quantum with a determinate amount that distinguishes it from others: each number is thus unique. The term "extensive magnitude" then names the unity – in the second sense just noted – that is common to all numbers. Such a magnitude is thus not only limited, but contains an amount – *just like* all the others. Note,

however, that what makes a quantum an extensive magnitude cannot be the specific amount it contains – 3, 4 or 5 – for that makes it a specific number. Such a magnitude must be defined, therefore, by the fact, common to all numbers, that it contains *an* amount at all. It must be characterized simply by having *some* amount.

Hegel goes on to argue that extensive magnitude is not purely extensive but converts itself logically into intensive magnitude, or degree. Note that extensive and intensive magnitude should not be confused with discrete and continuous magnitude. The latter are kinds of *quantity* and are logically prior to the idea of the quantum, whereas the former are kinds of *quantum*. For this reason Hegel sometimes refers to them – for example, in the title of 1.2.2.B – as extensive and intensive quantum, rather than magnitude (*Größe*).

FROM EXTENSIVE TO INTENSIVE MAGNITUDE

A *number* (*Zahl*), as we have seen, is a fully determinate quantum. It is not, however, a merely finite determinacy, but an infinitely self-relating determinacy: a determinate *one*. A finite something owes its determinacy partly to itself: it is what it is *in itself* (*an sich*). Yet it also owes its determinacy in part to other things. As a limited thing, for example, it gains its full determinacy though a contrast with another: a field is most clearly a field in *not* being a wood. It is also vulnerable, through its constitution, to being changed – and so given a new determinacy – by the other things to which it relates (see SL 96-9 / LS 120-3). A number, by contrast, is determinate purely within itself, or, as Hegel puts it, it is "absolutely" determinate within itself.[2] It enjoys this internal determinacy because it contains a definite amount of units, which is what it is regardless of the other numbers there may be. 3 certainly differs from 4 or 5; but even if there were – impossibly – no other numbers, 3 would be no less definitely 3. It is 3 because it contains precisely that amount of units, no more and no less. A number is thus a wholly self-relating one that contains an independently given determinacy, and as such a self-contained determinacy it is completely *indifferent* to every other number. Hegel pulls these thoughts together in the opening lines of 1.2.2.B.a.2 (also quoted on 2: 136):

> Determinacy through number – how big something is – does not require being distinguished from something else of magnitude, as if to the determinacy of one thing belonged its magnitude and that of another, for the determinacy of magnitude as such is a limit determinate for itself, indifferent and simply related to itself; and in number this limit is posited as enclosed in the one that is for itself and it has externality, the relation to another, *within itself*.
>
> —SL 182-3 / LS 231-2

A number, then, owes its determinacy solely to the amount within it; and insofar as it has an amount *within* it, it is a unity in the second sense mentioned above: one that contains (rather than differs from) the many ones. As such, however, as we have seen, it is not just a number but an extensive magnitude. Note again the subtle difference between these two categories. All numbers, as numbers, differ from one another; insofar as they are extensive magnitudes, however, they are all *alike* – the units within each of them form the same extensive unity.

As also indicated above, there is a further, related difference between a number and an extensive magnitude. A number as such contains a specific amount of units that gives it a unique determinacy. Yet, as an extensive magnitude that is just like all other numbers, it simply contains *some* amount that is not further specified. Such an amount does not, therefore, give it the unique determinacy it enjoys as a number. As Hegel puts it, in an extensive magnitude the amount, "*as an aggregate of numerical ones, does not constitute* the determinacy of number" (SL183 / LS 232).

It is important to note that an extensive magnitude does not lack an amount altogether.[3] In the quantum prior to its being a number, the plurality it contains is not yet the ground of its determinacy. Accordingly, the plurality is not itself determinate or an *amount*. In the number, by contrast, the plurality *is* an amount. In the extensive magnitude, it is still an amount; but it is equally just *some* amount, not a specific one. In other words, the amount is present in a negated form (as what Hegel goes on to call a "*sublated* amount" [SL 183 / LS 233]). An extensive magnitude, therefore, does not lack an amount (like the simple quantum), but it contains an amount that is *no longer* an amount in the fully determinate sense.[4]

An amount that is not completely determinate, however, is not actually an *amount* at all, for the latter is by definition completely determinate: it is *3* or *4* or *5*, and so on. An amount that is not determinate in this way is in fact merely an indeterminate *aggregate* (*Menge*) or *plurality* (*Vieles*). As we will now see, this dramatically transforms the extensive magnitude.

As we know, a number contains just two logical components: its amount and its unity (see SL 169 / LS 214). The extensive magnitude, as the form common to numbers, is thus also restricted to these two components. If, therefore, the many in the extensive magnitude do not constitute a definite *amount*, they must – by virtue of their homogeneity – constitute its *unity*. Yet the unity of a number, insofar as it differs from its amount, is qualitatively distinct from the latter: the amount is an explicit plurality without continuity and unity, whereas the unity is one without explicit plurality.[5] This must remain true in the extensive magnitude. The unity formed by the many ones in this magnitude must, therefore, be one without explicit plurality – that is, without inner extension – and so be a "simple unity" (SL 183 / LS 232).

This is a highly paradoxical result, but it follows from the logical structure of the number and the extensive magnitude. The many ones in a number constitute both its amount and its unity, but they constitute the one only insofar as they do *not* constitute the other. The amount and the unity remain distinct in the extensive magnitude. In that magnitude, however, there is no longer an amount as such, but merely an aggregate of homogeneous units. Since these units do not form an amount, they must, therefore, form a unity that is qualitatively distinct from the latter, namely a simple unity without explicit plurality. Accordingly, the many in the extensive magnitude do not form a unity of *many* ones, but "collapse" into, and disappear in, the unity they form (SL 183 / LS 232).[6]

Extensive magnitude proves, therefore, to be a unity in which the discontinuous externality of the ones is explicitly *negated*; or, as Hegel puts it, "the externality that constituted the ones of plurality *vanishes* in the one as the relation of number to itself" (SL 183 / LS 232, emphasis added). Since the many ones in extensive magnitude cease in this way being many and come to coincide completely with the numerical unity within which they are "enclosed" (*eingeschlossen*), they are, so to speak, absorbed by the unity to which they belong. Note that the magnitude does not thereby revert to being a bare quantum, for the latter, though not yet a number, encompasses *many* ones. Extensive magnitude becomes a new unity, in which there is no longer any explicit plurality or sum of mutually external units. Such externality, however, is what makes extensive magnitude *extensive*: numbers are extensive magnitudes precisely because they contain units that are external to one another. If such mutual externality vanishes in the unity of the magnitude, then the extensive character of that magnitude must vanish, too. Extensive magnitude thus proves, through its own inherent logic, to be a magnitude that is *not* extensive after all, but an *intensive* magnitude. The latter is what we also call a degree (*Grad*). Quantity, therefore, must come in degrees as well as sums or amounts.

Speculative logic has led us to an important insight: extensive magnitude turns, of its own accord, into intensive magnitude. These two are not just distinct kinds of quantum, as they are for Kant, but the former makes the latter necessary.[7] According to Winfield, it does so for the following reason: the many in extensive magnitude are all *alike* and so constitute a simple unity, which is the degree. In Winfield's words:

> Every extensive magnitude converts itself into an intensive magnitude *once you recognize that its units are one and the same*. In a sense, the plurality gets collapsed, and one has the amount without having to deal with the plurality. Degree immediately presents the determinate magnitude, canceling out the discrete aspect. No plurality lies within degree.[8]

What Winfield says in these particular lines, however, can only be part of the story, for it leaves us with a problem: the many in an extensive magnitude may well form a unity, but why should that unity lack explicit plurality? Why should it be an utterly *simple* unity in which the discrete aspect, as Winfield puts it, has been "cancelled"? As we have seen before in our account of quantity, many ones can constitute a unity while remaining many: they can form a unity *of many ones*. This is precisely what occurs in discrete magnitude, as it gives rise to the quantum: the many ones in such magnitude form a unity because they are "constant", but the unity they form consists in "self-continuing as such *in the discreteness of the ones*" (SL 167 / LS 211, emphasis added). If this is the case at that point in the development of quantity, then why not here too? No answer to this question is to be found in the lines from Winfield just cited: for the simple fact that the ones in an extensive magnitude form a unity does not by itself mean that the unity must lack explicit plurality.

Hegel's own text, however, points toward the answer. It does so by reminding us, through a subtle linguistic contrast, that the units in an extensive magnitude form a unity precisely insofar as they do *not* constitute a determinate amount (which is no longer present in such a magnitude). For this reason the unity they form must lack the explicit plurality that belongs to an amount. Hegel writes:

> Further, this many of the limit itself is, like the many as such, not unequal within itself but continuous; each of the many is what the other is; consequently, the many as plural being-outside-of-one-another, or as discrete, *does not constitute determinacy as such*. This many thus collapses for itself into its continuity and becomes simple unity.
>
> —SL 183 / LS 232, emphasis added[9]

After a dash in the text Hegel repeats the point, making the contrast slightly stronger. He refers just to "number", but he is talking specifically about extensive magnitude:

> Amount is only a moment of number, but, *as an aggregate of numerical ones, does not constitute* the determinacy of number; on the contrary [*sondern*], these ones as indifferent and self-external are sublated in number's return into itself.

The contrast contained in these last lines – "does not" / "on the contrary" – clearly suggests the thought I outlined above: the many ones in the extensive magnitude no longer form a determinate plurality or amount and so constitute *by contrast* a unity without explicit plurality. The magnitude must thus be an intensive, rather than an extensive, unity.

What converts extensive into intensive magnitude, therefore, is not just the fact that the many in it are homogenous and so constitute a unity. It is the fact that the many constitute a unity, *as opposed to an amount*. To put the point another way, in the extensive magnitude the amount negates itself into its own opposite. An amount is the fully determinate plurality that defines a number. In the extensive magnitude, however, it is no longer fully determinate, but just some unspecified amount. As such, it is in fact no longer an *amount* at all, but merely an indeterminate aggregate or plurality. Since this plurality no longer constitutes an amount, however, it must constitute the *unity* of the magnitude, for amount and unity are the only components of that magnitude. In so doing, it constitutes a unity that is quite distinct from, and lacks, explicit plurality itself, and so is an utterly "simple unity". The many in the extensive magnitude thus "collapse" into a unity without inner plurality or extension, and constitute a magnitude that is a simple, self-relating one. In Hegel's words, "this externality, as plurality as such, collapses into undifferentiatedness and sublates itself [*hebt sich auf*] in the one of number [*in dem Eins der Zahl*], in its self-relation" (SL 184 / LS 233). This simple, self-relating magnitude is the degree.

A degree, however, is what extensive magnitude, and so number, and so the quantum, proves to be, and so is still a limited, *determinate* unit of quantity. Yet it contains no sum or set of given units within itself and so, as well as being a simple unity, is a "*simple determinacy*" (*einfache Bestimmtheit*) (SL 183 / LS 232). The complete determinacy of a quantum, however, is made explicit, or "posited", in a number (see SL 169 / LS 213). A degree, like an extensive magnitude, must therefore be "expressed by a *number*". Yet such a number will differ from the numbers we have already encountered, for it will have to match the simplicity of the degree. It will, indeed, contain the specific determinacy that belongs to a specific amount – because only in that way can it render the degree fully determinate – but it will not contain the amount as such. It will thus not be the *sum* of separate units, but will be a wholly simple number. Such a number will not be 3, 4 or 5, but the *3rd*, *4th* or *5th*.

This is not to deny that we can speak of the temperature, for example, being "10 degrees". To think of this number explicitly *as* an intensive magnitude, *as* a degree, however, is to think of it, not in this way, but as the *10th* degree on the temperature scale. If we think of the number like this, it will denote, not a sum of units, but *one* simple quantum that differs from all others. It will not encompass 10 "units" (of heat), as we might gather apples, but will indicate rather *the* specific degree that the temperature has reached, namely the 10th. If, on the other hand, we think of "10 degrees" as a sum, then we are thinking of it as an extensive, not as an intensive, magnitude. In Hegel's own words, "when we speak of 10 or 20 degrees, the quantum which has that many degrees is the tenth or twentieth degree, not the amount or sum of them; as such, it would be an extensive quantum; but it is only one degree, the tenth or twentieth" (SL 183 / LS 232-3).

Speculative logic demonstrates, therefore, not only that quantity must take the form of degrees, as well as amounts, but also that numbers must take two forms. They must be cardinal numbers – 3, 4 and 5 – and ordinal numbers – 3rd, 4th and 5th. The distinction between these two types of number, together with that between extensive and intensive magnitude, is built into the very fabric of quantity, and of being, itself. Furthermore, every cardinal number, as an extensive magnitude, must itself be intensive and thus an ordinal number (because extensive magnitude does not just have intensive magnitude as its counterpart but converts *itself* into the latter). This does not mean that 3 is always and only the 3rd – it may be 1st, 2nd or 3rd in a series – but it has to have some ordinality. It is not, however, the logic of number as such that makes ordinal, as well as cardinal, numbers necessary, but the logic of extensive magnitude. In the latter the amount is reduced to an unspecified aggregate of homogeneous units. This homogeneity gives the magnitude no specific determinacy but, *on the contrary*, constitutes its unity. Since the homogeneity of the units forms a unity as opposed to a determinate amount, the unity is itself simple and without explicit plurality. As such a unity, the magnitude is intensive, or a degree. This degree in turn finds expression as a number, and so numbers must be ordinal, as well as cardinal.

INTENSIVE MAGNITUDE

Extensive magnitude contains a sum of units that are external to one another, whereas intensive magnitude contains no such sum and is a simple, though determinate, unity. An intensive magnitude or degree is thus not a quantum of *discrete* magnitude, as we encountered it above, since the latter contains an aggregate or sum of units and so is extensive in character (even though, unlike number, it does not as such comprise units that are wholly discrete [see 2: 4-5, 15]). Yet nor is a degree an instance of *continuous* magnitude, such as we see in space and time. This latter magnitude, though explicitly continuous, is divisible into discrete units, such as "heres" and "nows", and so, implicitly at least, is extensive, rather than intensive. A degree, by contrast, is the opposite of an extensive magnitude: it is a wholly *simple* unity. Not only is it not an aggregate or sum, therefore, but it is not divisible into discrete units either. As such, it is a new kind of quantity altogether (see 2: 3-4).

Yet a degree remains a *quantitative* unity, or *continuity*, and so must contain *many* ones after all: the tenth or twentieth degree is "a quantum that has that many degrees" (SL 183 / LS 232). It includes many, however, not *as many*, nor even as the "real possibility" of division, but solely *as fused into one*. A degree is thus "not an aggregate or several [*Menge oder Mehreres*] *within itself*", but is a single, *unified* plurality that Hegel calls a *Mehrheit* (translated by Miller as a "plurality only in principle").[10] This *Mehrheit* is notionally divisible, since it is,

indeed, a plurality; but it is not actually divisible, since it is a wholly simple unity. It is expressed, and rendered fully determinate, by an ordinal number, such as 3rd or 4th. Such a number is thus itself a simple unity that contains a many "in principle" but cannot actually be sub-divided. The 3rd contains 3, and the 4th 4, yet not as a sum but as that which gives determinacy to its simple indivisible unity.

An intensive magnitude thus lacks the plurality of mutually external ones that makes extensive magnitude extensive. Hegel argues, however, that a degree is nonetheless inseparable from a certain externality. A degree is a quantum in the form of a simple self-relating one or "being-for-self" and, as such, is completely self-contained; yet Hegel contends that it is also *not* self-contained but *external to itself* (SL 183-4 / LS 232-3). Every quantum, insofar as it is a one, is external to itself in the sense that it encounters further instances of itself – other *quanta* – outside itself. The degree is external to itself, however, for a more particular reason: it has its own *determinacy* outside itself.

As we have seen, the degree has a certain determinacy that is expressed in a number: it is the 10th or 20th degree. It owes this determinacy to a specific amount: a degree would not be the 10th, as opposed to the 20th, if it did not contain the number 10, rather than 20.[11] Yet unlike the extensive magnitude (before it mutates into the degree), the degree does not contain an amount as an aggregate or sum of units. It is thus contradictory, for it cannot be what it is without an amount and so is inseparable from the latter, yet it has no amount within it. Hegel concludes, therefore, that it must have the amount to which it owes its determinacy *outside itself*. In Hegel's words:

> Degree, therefore, which, as simple within itself, no longer *has* this *external otherness in it*, has it *outside* it and relates itself to it as to its determinacy. A plurality external to it constitutes the determinacy of the simple limit which the degree is for itself.
>
> —SL 184 / LS 233

The degree, as a quantum, relates to other degrees. It is external to itself, however, not just for that reason, but because the amount that grounds its *determinacy* lies outside it. An extensive magnitude, as a specific number, is what it is thanks to the specific amount of units it contains; a degree, by contrast, is what it is thanks to the amount of *other* degrees there are. Now in the sphere of quality something is, of course, also partly determined by the other things to which it relates: its constitution, for example, is vulnerable to being changed by those other things. A degree, however, has its determinacy wholly outside it: what it is depends completely on the amount of other degrees to which it is related. A degree is thus necessarily one "*among* a *plurality* of such intensities" (SL 184 / LS 233), not just because every one (*Eins*) is one of many, but because it is only in this way that it can be the *determinate* one, or quantum, that it is.

This is not to deny that each degree, as a one and a quantum, is wholly self-related – indeed, is a completely "simple self-relation" – and in this respect is indifferent to all the others (SL 184 / LS 233-4). Each degree is itself and, as an ordinal number, has its own identity and determinacy: it is the 18th *or* the 19th *or* the 20th. And yet the degree enjoys its identity thanks only to the *other* degrees outside it. In its separateness and indifference, therefore, the degree is intrinsically *related* to those other degrees. Hegel sets out the contradictory character of the degree as follows:

> As a self-relating quantitative determination, each degree is indifferent to the others; but it is just as much related in itself [*an sich*] to this externality, it is only through the latter that it is what it is; its relation to itself is, in short, the non-indifferent relation to what is external, and in this relation it has its quality.
>
> —SL 184 / LS 234

None of this means, by the way, that the 20th degree is the 20th because it has *20* degrees outside it. Some of Hegel's language might lead one to draw this conclusion. He writes, for example, that the amount, which was contained within the extensive magnitude, is now "posited outside the number", so one might be forgiven for thinking that the 20 units in the number 20 are simply pushed outside the number when it proves to be a degree (SL 184 / LS 233). This interpretation, however, overlooks the fact that the degree as an ordinal number, just like the cardinal number, has its *own* identity and determinacy: the number 20 belongs to the 20th degree *itself*.[12] Hegel's point is thus not that there are 20 other degrees outside this one, but that the amount of other degrees determines this one *to be the 20th*. The latter owes its own determinacy *as* the 20th to the *others* outside it and has its determinacy only in *relation* (*Beziehung*) to those others. It is the 20th quite simply because there are 19 before it.

There is a further point to be made about intensive magnitude. Since the many degrees are inherently related in the way we have described, they are not purely external to one another but form an explicit *continuity*.[13] This continuity, however, differs from any we have encountered so far. Previous continuities have arisen because the units concerned are all the *same* in being discrete. Degrees are, of course, also all the same, insofar as we disregard their specific ordinal numbers and consider them simply as intensive magnitudes. They form a continuity, however, not because they are the same, but because they are different but inseparably *related* quanta. As Hegel puts it, they form "a continuous progress, a flow, which is an uninterrupted, indivisible alteration" (SL 184 / LS 234). This new continuity involves an "alteration" because degrees, expressed as ordinal numbers, are explicitly different from one another and their continuity is thus the transition from one degree to a different degree.

Their continuity, therefore, takes the form of a rising and ascending "scale of degrees", and a degree owes its determinacy to its place on that scale.[14]

THE IDENTITY OF EXTENSIVE AND INTENSIVE MAGNITUDE

Having shown that extensive magnitude mutates logically into intensive magnitude or degree, Hegel now shows that the development also goes in the other direction. This is due to the twofold character of the degree's determinacy.

On the one hand, the degree has its determinacy through its relation to *other* degrees: its place in the overall scale of degrees makes it the 19th or 20th. Each degree occupies a unique place in that scale and so differs from the others; yet it owes its identity *to* those others. As Hegel writes: intensive magnitude "is determined by *other* intensive quanta and is continuous with its otherness, so that its determinacy consists in this relation [*Beziehung*] to the latter" (SL 185 / LS 234).

On the other hand, each degree, or intensive magnitude, is "determinate in its own self" (*an ihr selbst bestimmt)* (SL 185 / LS 235).[15] Although its identity is determined by the amount of degrees outside it, that identity is specific to it: it is itself the 19th or the 20th degree. It is true that all degrees are alike in being degrees. Each, however, has its *own* determinacy, expressed by an ordinal number. Each is thus determinate *within itself*.

As we have seen, a quantum, as a number, derives its determinacy from its amount, and as an extensive magnitude it contains its amount within itself: it encompasses a sum of units. A degree is the negation of an extensive magnitude, but it, too, owes its determinacy to an amount: the 20th degree would not be the 20th without 20 (rather than 19) (see SL 185 / LS 235). Since, however, the degree is a simple, unextended unity, it does not encompass a sum of units *within* itself. The amount that determines its identity thus has to fall outside it – though this means, not that the 20th has 20 degrees outside it, but that the amount of other degrees determine it to be the 20th.

Yet, as we have noted, the degree is not only determined by other degrees, but also determinate in itself: the 20th degree is *itself* the 20th. Since the amount, 20, is inseparable from being the 20th, that amount must also belong *to* the degree. It cannot simply fall outside the degree, therefore, but must be "*its* amount". Accordingly, as Hegel writes, "the twentieth degree contains the twenty *within itself* [*an ihm selbst*]" (SL 185 / LS 235, emphasis added). This, however, significantly changes the degree: for insofar as the latter contains an amount or sum within itself, it is no longer an intensive but rather an extensive magnitude. As Hegel puts it, "insofar as the amount is its own, and the determinacy is at the same time essentially as amount, the degree is extensive quantum".

The extensive and intensive magnitude thus mutate logically into one another. The units in an extensive magnitude are external to one another. Insofar as they form a *unity* as opposed to an amount, however, they cease being mutually external and the magnitude ceases being extensive: it proves to be an intensive magnitude or degree. Conversely, the degree has its determinacy in an amount that does not just fall outside it but is also its *own*, and so lies *within* it, and the degree thereby proves to be an extensive magnitude. Every cardinal number can thus be expressed as an ordinal number, and vice versa. This feature of numbers is not just a product of convention but is made necessary by the nature of quantity, and ultimately of being, itself.

Note that as the one kind of magnitude becomes the other, the determinacy involved is unaffected: 20 remains the same, whether it is expressed as an extensive or intensive magnitude. As Hegel remarks later in the text, "the difference between extensive and intensive quantum is indifferent to the determinacy as such of the quantum" (SL 189 / LS 239). A quantum is rendered fully determinate in a number, and the latter in turn proves to be both an extensive and an intensive magnitude, or a cardinal and an ordinal number. The two magnitudes are thus not wholly distinct from the number, but are the two forms taken *by* the latter. Accordingly, the number's determinacy remains the same in either form: extensive and intensive magnitude are "one and the same determinacy of the quantum" (SL 185 / LS 235). This is not, of course, to deny that numbers can have different values from degrees. Hegel's point, however, is that one numerical value or determinacy must take both an extensive and an intensive form: 3 is not only 3 but also the 3rd number (though we could start from it and so make it the 1st).[16]

Moreover, not only do both magnitudes preserve the same determinacy as they mutate into one another, but their dialectic also brings them together to form a *single* magnitude or quantum. In that dialectic extensive and intensive magnitude initially remain distinct, since each turns into *the other*. Yet precisely because each proves *to be* the other, the two cease simply being distinct and form an indissoluble *unity* or "identity": a single quantum that is both extensive and intensive.[17] In this way, the unity that is initially enjoyed by the quantum (in 1.2.1.C) is restored. When it first arises, the quantum is a determinate unity or unit. This unit then gives rise logically to number, which turns out to have two different forms: extensive and intensive. When these then pass over into one another and together form one being, the quantum is once again a unity. The quantum is thus in truth a single self-relating unity or unit that is both intensive and extensive and so has an ordinal and a cardinal number. Hegel sees a concrete example of this unity of intensive and extensive magnitude in the fact that an increase in temperature also entails the physical expansion of the substance concerned. As he writes:

> *heat* has a *degree*; this degree, whether the 10th, the 20th, etc., is a simple sensation [*Empfindung*], something subjective. But this degree is equally present as *extensive* magnitude, in the form of the expansion of some fluid matter, of mercury in the thermometer, of air, clay [*Ton*], etc. A higher degree of temperature finds expression in a longer mercury column, or in a narrower clay cylinder [*Tonzylinder*]; it warms up a larger space in the same way as a lower degree warms up a smaller.
>
> —SL 187-8 / LS 238[18]

We have not, however, reached the end of the story of extensive and intensive magnitude, for Hegel argues that the unity of the two yields more than a unified *quantum*. This quantum, as a self-relating unit, is, of course, a *something* (*Etwas*).[19] The result of the dialectic of the two magnitudes is thus "*something*" that "is a quantum" (SL 185 / LS 235). Yet this something cannot just be a quantum, as we shall now see.

Recall that extensive and intensive magnitude form a unity – a "something" – by ceasing to be two quite distinct magnitudes, that is, by *negating* the difference between them. As Hegel writes, the unity they constitute is "self-relating through the *negation of its differences*" (SL 185 / LS 235). The difference between the magnitudes is, however, inherent in the nature of number, the quantum and quantity itself. A number may retain its determinacy as both magnitudes, but, as we have seen, it *must take the form of that difference*: a number, which is a fully determinate quantum, and thus fully determinate quantity, must be both extensive and intensive, both a cardinal and an ordinal number. This, I take it, is what Hegel has in mind when he states that "these differences [. . .] constitute quantitative determinacy in its determinate being [*die daseiende Größe-Bestimmtheit*]".

The something that emerges at this point in the *Logic* must, therefore, be the negation, not just of the difference between extensive and intensive magnitude, but thereby also of the fully determinate quantum to which that difference belongs. Indeed, it must be the negation of quantity as such, since the fully determinate quantum is simply what quantity itself proves to be. As a self-relating *something*, however, it must not only lack the difference proper to quantity, but must also be something *other* than quantity; moreover, as other than quantity, it must be indifferent to the latter. At this point, there is only one thing that can be indifferent to quantity in this way, namely *quality*, since that is the only other form of being to have emerged. The something that arises here must therefore be, not just a quantum, but also a "*qualitative something*" (SL 185 / LS 235).

By turning into one another and so forming a unity, therefore, extensive and intensive magnitude constitute "something" with a twofold character. On the one hand, this something is a *quantum* that retains its determinacy, whether it is extensive or intensive; on the other hand, it is a *qualitative* something that is

indifferent to all determinate quantity, whether extensive or intensive. In Hegel's words, "*something* is a quantum, but now qualitative determinate being [*Dasein*], as it is in itself, is *posited* as indifferent to it" – as indifferent to the quantum that *it* is.

Note that the newly derived something must be both quantitative and qualitative because of an ambiguity in the idea of the "negation" of the difference between extensive and intensive magnitude. Insofar as that difference alone is negated, what emerges is a quantum (or quantitative something) that is both extensive and intensive. Yet the negation of that difference is also the negation of the fully determinate quantum – and thus fully determinate quantity – to which the difference belongs; in that respect, therefore, what emerges through the negation must be a something that is no longer quantitative but qualitative. These somethings, however, are not separate from one another: they are not two different things. The qualitative something that emerges here is itself the quantum that is both extensive and intensive. Or, to be more precise, the something, as qualitative, *has* both an extensive and intensive quantum, since it is indifferent to the quantum it is.

Before this point in the *Logic*, Hegel, notes, the quantum and number – indeed, all forms of quantity – are thought "without a something that would be their substrate" (SL 185 / LS 235). Quality is shown to lead logically to quantity, but quantity, as it first appears, stands alone without belonging to any underlying qualitative being. Now, however, we see that the difference inherent in the quantum, through its collapse into a unity, makes quality necessary again in the form of a *something* – a something that also is, or has, an extensive and intensive quantum, but that, *as qualitative*, is indifferent to being, or having, either. As Hegel himself puts it, "positing" the quantum's determinacy in its "differences [*Unterschieden*] as extensive and intensive quantum is the return into this unity which, as negative, is the something posited as indifferent to them" (SL 186 / LS 235-6).

This, of course, is not the first time that quantity has generated *something*. This occurred when the very idea of a quantum arose. Furthermore, a quantum is a matter of indifference to quantity, insofar as no particular quantum is inherent in quantity itself. Yet quantity and the quantum are not just indifferent to one another, since the quantum as such, if not any particular quantum, is the limited unit that is made necessary by quantity: it is the determinate form that quantity itself must take. What has now emerged is a something that is *indifferent* to quantity in all its developed determinacy. This something is indifferent to the determinate difference between types of quantum, and so to the quantum as such, and so to quantity as such. Accordingly, it must itself be *qualitative*, rather than quantitative. This is the point, therefore, at which it becomes clear that, and why, quantity is not a free-floating form of being, but one that is inseparable from something qualitative that is indifferent to it.

Note, however, the ambiguity in this last idea. First, quantity is inseparable from quality, since it makes the latter necessary. Quality, therefore, cannot itself be utterly indifferent to being quantitative: quantity proves to be attached to the qualitative something – to be the quantity *of* that something – and so the latter is itself shown to be, necessarily, the bearer of quantity. Note that quantity here makes necessary, not some new, hitherto unknown qualitative something, but simply *the* qualitative something as such, and it shows itself to be inseparable from the latter. For this reason, the qualitative something itself is shown at this point to be inseparable from quantity. Yet, second, this something, as *qualitative*, is also indifferent to quantity and so remains unaffected by its magnitude or degree; that is to say, it is indifferent to the particular quantum that attaches to it – unless that quantum marks the limit that a thing may not exceed if it is to be what it is, and so constitutes the thing's measure (*Maß*).

In a remark at the start of the section on quantity Hegel maintains that something remains what it is, and so retains its quality, if it changes its quantitative limit – increases or decreases its size or degree – whereas it becomes something different if it changes the quality that defines it (SL 153 / LS 193). In this sense, a thing's quality is indifferent to its quantity: a field remains a field if it gets bigger or smaller, and a colour remains that colour whether it gets brighter or paler (albeit within limits set by its measure). The idea that quantity belongs to something qualitative which is indifferent to its quantity is, however, asserted by Hegel rather than proven. Indifference is, indeed, shown to be a necessary logical feature of quantity itself: quantity is the unity formed by the one that continues beyond itself and so is indifferent to its own limit. Yet this does not prove that quantity is attached to, while being a matter of indifference to, a *qualitative* thing; on the contrary, quantity initially supplants quality (and stands on its own), even though it is made necessary by the latter.[20] The attachment of quantity to something qualitative is proven, however, by the dialectic of extensive and intensive magnitude: for, as we have just seen, these two magnitudes collapse into a self-relating unity that is both a quantum *and* a qualitative something that is indifferent to being a quantum and to the determinacy of the latter. From this point, therefore, we know that, and why, quanta do not exist on their own, or just alongside qualitative things, but are determinations *of* such things themselves. (So, in the example above, a degree is not just a quantum in the abstract, but the temperature of *mercury* or *air*.)

THE CHANGING QUANTUM

Later in the *Logic* quantity will constitute quality that is itself identical to quantity. Such quality will consist in the *self-relating* of the quantum that is exhibited when the latter raises itself to a power of itself (see SL 279 / LS 361). We have not yet, however, been led to that idea by the logic of quantity (though

Hegel briefly discusses "raising-to-a-power" in the remark on counting and addition). At this point in the *Logic*, quality emerges in the form of something (*Etwas*) that is the indifferent "substrate" of quantity. Yet precisely because this something is indifferent to quantity, it does not enter into the ongoing dialectic of quantity itself.[21] Our focus, therefore, must now be on the unified *quantum* that is restored by the dialectic of extensive and intensive magnitude. What we learn is that such a quantum is subject to *change* – change that turns out to belong necessarily to the "quality" of being such a quantum.[22]

Hegel's explicit derivation of quantitative change is, however, problematic. He first reminds us that a degree has "its determinacy not in it but in another quantum" (though, strictly speaking, it has its determinacy through the *many* others that lie outside it) (SL 184, 189 / LS 233, 240). He then moves straight to the idea that a quantum is "posited in absolute continuity with its externality, with its otherness". This latter idea then justifies the claim that the quantum must *change* into another quantum in order to be what it is (SL 189 / LS 240). This sequence of thoughts suggests that the logical structure of the degree alone makes quantitative change necessary. Hegel does not say as much explicitly, but the implication is there (and in the first edition of the *Logic* this connection is made explicit).[23]

There is, however, a problem with this derivation of change. A degree is determined *by* other quanta, that is, by its place in the scale of quanta, but it is not obvious why this fact alone should require the degree to change *into* another degree. The degree will certainly be changed if the number of quanta outside it changes; but why should the simple fact that a degree is determined by 19 other quanta to be the 20th require it to change into the 19th or the 21st? As I see it, there is no good reason, and there is thus nothing in a degree itself – nothing in the relation that makes it *this* degree, rather than that – that explains why it must change.

So why does the quantum have to change? It must do so, I would suggest, not simply because it is a degree, but because it has proven to be the *unity* of intensive and extensive magnitude. Hegel does not say this explicitly himself, but this suggestion, in my view, makes sense of what is otherwise a rather obscure transition. The logic of quantity requires the quantum to become an extensive and then an intensive magnitude, but it then makes the unity of these two magnitudes necessary – a unity that takes the form of a newly unified quantum that we might call "concrete". Any further development should, therefore, be grounded in this "concrete" quantum.

Since this quantum is both extensive and intensive, it is *reducible to neither*, and in my view it is specifically this feature of the quantum – when rendered fully explicit – that makes it necessary for the quantum to change. The reason why is as follows. Insofar as the concrete quantum is intensive, its determinacy is "the negation of itself": that determinacy lies "not in it but in another

quantum" (SL 189 / LS 240). Insofar as the quantum is extensive, however, its determinacy resides in an amount that it contains within itself – an amount that is not determined by other quanta outside it (as in the simple degree) but that belongs to the quantum itself. Yet the quantum cannot have its determinacy solely within itself, precisely because, as just noted, it also has the latter in *another* quantum. The quantum's determinacy is thus contradictory. It is, on the one hand, the quantum's *own* determinacy (and so is not given to the quantum by others); yet it does not belong to the quantum alone, but is also to be found outside the latter. The determinacy thus *continues*, beyond the quantum to which it belongs, in another quantum. This fact – not just the simple structure of the degree – is what justifies Hegel's claim that the quantum is "posited in absolute continuity with its externality, with its otherness".

Now continuity in turn can be understood in two ways. A thing can continue beyond itself by remaining *itself* in another or by becoming *another*. The second sense of continuity is at play here. The concrete quantum certainly has its own determinacy, but the latter also lies beyond it in *another* quantum that is *not* the same as it. The quantum's determinacy thus continues beyond it, not by remaining itself, but by becoming a different one. Such continuity is a necessary feature of the newly unified quantum, rooted in the fact that the latter is both extensive and intensive, and so reducible to neither. As Hegel puts it, "the quantitative determinacy continues itself into its otherness in such a way that it has its being [*Sein*] *only* in this continuity with an other" (SL 189 / LS 240, emphasis added). In order to be itself and to enjoy its determinacy, therefore, the quantum cannot simply be what *it* is; it must also go beyond itself and become *another* quantum. In other words, it must *change* into another quantum and so increase or decrease itself.

Hegel insists that a concrete quantum not only can change into another but *must* do so because its determinacy extends, by its nature, into another. Such a quantum is not just subject to possible alteration by external factors, but it changes, and must change, of its own accord. Its determinacy is thus not a fixed limit but an intrinsically varying limit, or "a limit that becomes" (*eine werdende Grenze*). We noted earlier that numbers can be added to or subtracted from one another externally (see 2: 27, 31). Now we learn that quanta that are expressed by both cardinal and ordinal numbers are inherently changeable themselves. Quantity, like quality, harbours its own intrinsic principle of change.

Note, however, that the ground of change in quantity differs subtly from the ground of change in quality. In the sphere of quality, something changes simply because it becomes *other* than itself, either through itself or through the influence of something else (see SL 92, 97 / LS 114, 121). The something also preserves its identity in changing (provided it does not go beyond its qualitative limit) and in that sense may be said to "continue" being itself in becoming other; such "continuity", however, is the result, not the ground, of the thing's

change. In the sphere of quantity, by contrast, a quantum changes *because* it continues beyond itself in another quantum. More specifically, its *determinacy* continues beyond it in another quantum. This moment of determinacy also distinguishes the continuity of the quantum from that of the simple one.

As Hegel points out, the simple one (*Eins*) also continues beyond itself: it repels itself from itself as many other ones. In so doing, however, the one does not become anything other than a bare one: it does not *change* into anything else. Its repulsion of itself is "the generation of that which is the same as itself", and so in continuing beyond itself the one remains "infinitely" related to *itself*. The quantum differs from the simple one, however, by being an explicitly "*determinate* one". It continues beyond itself, therefore, in another quantum that is also explicitly determinate. Now it is true that all quanta, as extensive and intensive, are alike. Each, however, has its own distinctive determinacy. In continuing beyond itself in another quantum, therefore, the quantum does not simply remain itself but becomes a quantum with a *different* determinacy. That is to say, it continues beyond itself by *changing* into a new quantum (SL 189 / LS 240).

What arises with the changing quantum is thus a new sense of continuity. The simple one continues beyond itself by remaining itself in every other one. The quantum, by contrast, continues beyond itself by changing into something different. It does not just preserve its determinacy beyond itself, therefore, but its determinacy takes on a new form. The quantum thus combines the continuity characteristic of the infinitely self-relating one with the process of becoming a different thing that finite things undergo. Indeed, this is what we should have expected: for the quantum is the explicitly determinate, limited one, and in that sense a *finite infinite*. As such, it is neither purely finite, nor a pure one.

Like the finite something (and unlike the one), the changing quantum becomes *something else*. The finite thing does so, however, by reaching its limit and ceasing to be itself. It does not, therefore, continue beyond its limit but ends there. The quantum, by contrast – like the pure one – does continue beyond itself and its limit. In becoming another quantum, therefore, the quantum does not simply cease to be, but is *itself* further determined: it is *itself increased or decreased*. It is true that in one sense finite things also continue beyond their limit: for, when a ram is sacrificed, the ashes are what the ram in dying has itself become (see 1: 229-30). Yet the ashes arise only because the ram *ceases* to be, and in that sense they replace it. Quanta, by contrast, do not simply replace one another, but form an explicit continuity by increasing or decreasing *themselves*.

As Paul Owen Johnson notes, it is at this point in the *Logic* that the mathematical definition of quantity receives its justification.[24] According to Hegel, mathematicians usually define quantity, or magnitude, as "that which can be *increased* or *decreased*" (SL 153 / LS 193-4). Such a definition is,

however, circular and "awkward" because "to increase" and "to decrease" mean to increase or decrease *magnitude* and so the latter remains undefined (see 1: 299-300). Hegel avoids such circularity, as we have seen, by deriving quantity and magnitude from quality in which nothing of quantity is already presupposed. His presuppositionless account of quantity has, however, now led to the very insight proclaimed by the mathematicians: the quantum is precisely that which can, indeed must, be *changed*, which is to say increased or decreased. This is a good example of the way in which, by initially setting aside a familiar assumption, Hegel is able to provide a presuppositionless and thus genuinely necessary justification of that assumption.

It has to be said that Hegel's text does not explicitly confirm the derivation of quantitative change that I have just set out. Moreover, as noted above, his account of that derivation in the first edition of the *Logic* begins directly from the degree. I explained above, however, why I regard it as problematic to move straight from the degree to change: the fact that a degree is determined *by* others to be the degree it is does not itself mean that it has to change *into* another degree (though it can, of course, become a different degree if the amount of other degrees in the scale changes). Furthermore, the development of quantity we have traced takes us not just to the degree, but to the unity of extensive and intensive magnitude. Any further development must, therefore, be generated by that unity.

On my reading, therefore, a quantum changes not just because it is a degree, but because it is a "concrete" quantum that is both extensive and intensive. There is nothing about being the 3rd or 4th degree *as such* that requires it to change: each degree simply is what it is, being determined to be such by the others outside it. Similarly, there is nothing about a particular cardinal number *as such* that requires it to change (though this would be true even if change were derived directly from the degree). The number 3 is defined by containing three units, no more and no less, and those three units will never make it anything other than 3: they will never make it, or require it to turn into, 2 or 4. (This is why Plato can understand numbers to be, or at least to be derived from, unchanging forms [see 2: 21-3].) As we have seen, the units in numbers are indifferent to the numbers to which they belong and can thus be added to and subtracted from one another; so in that sense numbers as simple numbers can be changed. Yet this does not mean that they themselves have to change. The number 3, therefore, does not have to change simply by virtue of being 3 and containing 3 "indifferent" units, and the same is true of every other cardinal number. What has to change is the *concrete quantum* that is both extensive and intensive and whose determinacy is indifferent to being either (and that is also the quantum of something qualitative).

Hegel's account of quanta and numbers thus explains both why they are unchanging *and* why they change. A cardinal or ordinal number taken by itself

is an unchanging entity. 3 is simply 3 and nothing else, and the 3rd is just the 3rd; what it is to be either does not alter. Yet the concrete quantum that is expressed by both a cardinal and an ordinal number is inherently variable and, indeed, varying. It must change because, logically, its determinacy continues beyond it in other quanta. It is crucial, however, to understand precisely what this last phrase means. It does not mean that the quantum has a fixed determinacy that continues as the quantum becomes another: the quantum with a value of 3 does not *continue to be 3* in becoming 4 and 5. This would be impossible, since numbers exclude one another through their amounts: 4 and 5 are, by definition, no longer 3.[25] Moreover, the logic of the concrete quantum requires that its determinacy continue beyond itself precisely by not remaining what it is, but by becoming a *different* one. When the quantum's determinacy is 3, therefore, it also continues *beyond* 3 and becomes 4, 5 and so on; that is, it stops being 3 and becomes *another* number (though the new quantum that arises does not simply replace its predecessor, but remains in explicit continuity with it, since it is what its predecessor has *itself* become). The quantum thus has no fixed determinacy but one that constantly changes. This determinacy continues beyond itself, not because it retains a particular value while taking on others, but because it constantly takes on new values *beyond* the particular one it has. In this way, the concrete quantum itself constantly becomes a different quantum and changes. Quanta are always getting bigger or smaller, even if in nature this can sometimes take a long time to occur.[26]

The concrete quantum thereby proves to be indifferent to any and every numerical value. It is indifferent, therefore, not only to being extensive or intensive, but also to being any given number, or particular quantum, at all. The simple number 3 is not indifferent to being 3 but *is* precisely that. A concrete quantum, by contrast, is no more tied to 3 than to any other number, but necessarily changes from one to another. It is thus essentially indifferent to anything it is. Note, by the way, that Hegel is claiming here, not just that qualitative things must change their quanta along with other aspects of themselves, but that such *quanta themselves* must change into one another. Concrete quanta, as opposed to cardinal or ordinal numbers in the abstract, are inherently variable and varying: they grow and diminish by their nature. Such change is not made necessary by time, since time has not yet been derived, and will not be derived until the philosophy of nature begins. The change is made necessary by the purely logical structure of the concrete quantum.[27]

CONCLUSION

Quantity as such is being that continues beyond itself and so is external to itself. Initially, it consists in the self-externality of the bare, discrete one. Now the *quantum* has become explicitly external to itself, too. It is external to itself first

in being an intensive magnitude whose determinacy is determined by others outside it, and then in being a concrete quantum that finds its determinacy *in* another quantum and so changes *into* that other quantum. The varying of quantity is thus not an accidental feature of it, but is rooted in the self-externality that is constitutive of quantity itself.

Yet not only does the quantum, in its concrete form, change into another, but that other in turn becomes another, and so on *ad infinitum*. Every concrete quantum impels itself beyond itself and so changes into another quantum. Quanta, as they are now conceived, thus constitute an endlessly ascending and descending series. This takes us to the brink of a new determination, which will occupy us in the next chapter, namely quantitative *infinity*. As we will see, the measure of a thing sets a limit to how big or small the thing can become while retaining its defining quality. There may also be limits to the quantitative changes things can undergo that are imposed by their material nature (though these cannot be considered until the philosophy of nature). No limit is placed on quantitative change, however, by quantity itself. The change of concrete quanta (and their numbers) into one another is in principle endless or "infinite".[28]

CHAPTER SEVEN

Quantitative Infinity

In the sphere of quality every something is subject to change; indeed, even its intrinsic nature, or "being-in-itself", can be altered by the other things with which it interacts (SL 92, 97 / LS 114, 120-1). Yet change alone does not make something a completely different thing: something preserves its identity through simple change and remains the thing it is, and does so even if its intrinsic nature is altered. Something can thus change without thereby becoming something else. It becomes *another* thing only when change takes it beyond its defining limit and it ceases being what it is. It becomes another thing, therefore, not just by changing but when change causes it to end and so renders it *finite*.

The sphere of quantity, however, is the sphere not of qualitative somethings but of quanta. Unlike the something, the quantum is not subject to change purely within itself. This is partly because it is a one and the nature of the one is to be unchangeable, but also partly because each quantum *qua* number is defined by a fixed amount of units.[1] The quantum changes, therefore, only by becoming *another* quantum altogether. In this respect, change in the sphere of quantity mirrors the succession of *finite* things in the sphere of quality, rather than simple qualitative change. The latter, as just noted, preserves the identity of the something concerned; by contrast, quantitative change, much like the succession of finite things, involves one limited quantum turning into *another*, "*and so on to infinity*" (SL 190 / LS 240).[2]

Yet this does not render the quantum itself purely finite, for "it *continues* into its otherness" and as such is *infinitely* self-relating (SL 190 / LS 241). On the one hand, the indifferent being-for-self of the quantum continues in the new quantum that it becomes, since this new quantum is once again a *quantum*; on the other hand, the quantum's determinacy also continues (in a new form) in that new quantum (see 2: 154-5). The quantum is, indeed, finite in the sense that it has a limit (set by the amount it contains);[3] but it continues beyond its limit and in that sense is explicitly unending and infinite.

There is, therefore, a significant difference between a quantum that becomes another quantum and a finite something that becomes another finite thing. The finite thing becomes another by *ceasing* to be itself, by coming to an end: the ram is consumed by fire and leaves nothing but ash. The ash is, indeed, what the ram in dying has itself become; in that sense, the ram continues in the ash and the death of the ram gives rise to unending, infinite being (see 1: 229-30). It is by definitively *ending*, however, that the ram continues in another form: qualitatively infinite being arises through real, irreversible finitude. The quantum, by contrast, does not come to an end when it becomes another quantum, but it continues explicitly beyond its own limit. This moment of explicit continuity ensures that the succession of discrete quanta is the *changing* of quanta into one another – their increasing or decreasing – rather than just their ceasing to be. This is not to deny that something does, indeed, end when one quantum becomes another. What comes to an end when a quantum increases from 3 to 4, however, is not the quantum itself but its being *3*. The quantum continues in the bigger quantum that it becomes: it increases *itself*.

To sum up: quantitative change mirrors endless qualitative finitude, rather than qualitative change. Yet it is change, rather than a form of finitude, because quanta, though limited, are not finite in the full qualitative sense – are not consigned by their nature to sheer non-being – but are infinite in their very limitedness: they continue explicitly beyond their own limit in the others that they become. We now need to consider in more detail the contradictory character of the changing quantum.

THE QUANTITATIVE INFINITE PROGRESS

The concrete quantum must change: it must become another quantum with a different determinacy. Yet in so doing it continues into its other: it continues beyond its own limit and increases or decreases *itself*.[4] In the new quantum, therefore, the first relates infinitely to itself and so proves not just to be a *limited* entity; that is, it proves to be an *un*-limited entity. The new quantum is thus not just another quantum; it is also the negation of the limitedness of the first or, in Hegel's words, "the negative of it as limited". It is that in which the first attains its own "unlimitedness" (*Unbegrenztheit*) and *infinity* (SL 190 / LS 241). The quantum is, of course, infinitely self-relating by itself, since it is a one; yet it also has its determinacy in, and so becomes, *another*. We have now seen that the quantum is infinitely *self*-relating even in becoming another. Note that this infinity is not simple endlessness, but it is the infinity of self-relation. Yet it remains problematic: for the quantum proves to be infinite in becoming *another* quantum. The infinity of the quantum thus actually lies beyond it in the other that it becomes.

Insofar as the quantum's infinity lies beyond it, it is of course a "bad" infinity (SL 190, 192 / LS 241-2, 244). Yet the quantitative bad infinite differs subtly from its qualitative equivalent. The qualitative bad infinite is directly opposed to the finite: it is the "non-finite" (SL 110 / LS 138). By virtue of being the negation of the finite it proves to be a *finite* infinite itself; in so doing, however, it turns into that to which, as infinite, it is explicitly opposed. The quantitative infinite is different in that, from the start, it *coincides* explicitly with the finite and limited. This is because the quantum relates to itself and so attains its *infinity* in another *finite*, limited quantum. In the sphere of quantity, therefore, the infinite and the finite are no longer simply opposed, but each entails the other. The finite quantum is itself infinite, since it continues beyond itself; and that infinity in turn is attained in another finite quantum.[5]

The relation between the quantum and its infinity is a complex one: such infinity lies beyond the quantum itself, but in so doing it coincides with another *quantum*. This complex relation, Hegel maintains, now generates the quantitative infinite progress. This progress incorporates the process in which quanta change into one another, but it is driven forward by the distinctive character of the quantum's infinity.

Quanta are external to themselves and so continue *beyond* their limits in *other* quanta. Accordingly, a quantum relates to itself and thereby attains its infinity in its "beyond" (*Jenseits*) (SL 191 / LS 243). Yet, insofar as that beyond is *not* the quantum itself – is the negation or "*non-being*" (*Nichtsein*) of the latter – it is infinite in the bad sense: it is, precisely, infinity that lies *beyond* the quantum. Yet, as we said, the quantum *continues* beyond itself: "the quantum consists precisely in being the other of itself, external to itself". Since this is the case, what lies beyond the quantum must be itself once again, that is, an other that is identical to it in being a quantum. The infinity *beyond* the quantum thus necessarily coincides with another limited *quantum*. Indeed, it is the very presence of another quantum that enables the first to attain its infinity beyond itself. That first quantum is finite and limited; since, however, all quanta as such are the same, the quantum relates infinitely to itself in the other *quantum* that lies beyond it. (It also comes into a further form of its own determinacy in that other quantum.) In Hegel's words, therefore, "the beyond is in this way recalled from its flight and the infinite is attained" (SL 191 / LS 243).

Yet since the new quantum is itself a finite, limited unit of quantity, it has, and continues into, its own *beyond*. That beyond is thus where this quantum attains its infinity; but it takes the form of yet another quantum, which has its own beyond, and so on. As each new quantum arises, Hegel explains, it continues into its own beyond, "and as it thus repels itself into the beyond, so equally does the beyond perpetually become a quantum". In this way a quantitative infinite progress is generated that coincides with an endless increase and decrease of the initial quantum.

The *qualitative* infinite progress arises because the finite points beyond itself to an infinite that, as the negation of the finite, *turns into* a finite infinite (which points beyond itself . . .). By contrast, the quantitative infinite progress arises because the finite quantum has a beyond which is *at once* its own infinity *and* another finite quantum (with its beyond). The qualitative difference between the finite and infinite is preserved in the quantitative progress, since the quantum's infinity lies beyond, and is the negation of, the quantum itself. Yet the two progresses are not identical because quantitative infinity, unlike the qualitative one, also coincides from the outset with what is limited and finite. (This mirrors the relation between continuity and discreteness at the start of quantity, each of which also incorporates its other explicitly within itself.) Hegel sums up the difference, but also the intimate connection, between the quantitative and qualitative infinite progress in these lines: "it is true that in the progress of the quantitative that to which the advance is made is not an abstract other as such, but a quantum posited as different; but this remains in the same way opposed to its negation" (and so is the infinity beyond, and qualitatively distinct from, the first quantum) (SL 192 / LS 244).

Hegel notes that the coincidence of infinity with the finite quantum is the ground of the familiar idea of a quantum that is infinitely big or infinitely small (SL 192 / LS 243). This idea is not a further logical determination made necessary by the quantum itself, but one that we are *misled* into formulating by the concepts that have now been made necessary. An infinitely big or small quantum is – or rather is supposed to be – infinite in the sense that it does not lead beyond itself to any further quantum but is self-contained or "for itself". In other words, it is unsurpassable. Yet it is also determinate and so contains its own distinctive amount. Its determinacy is thus (supposedly) the unsurpassable "absolute determinacy" (*absolute Bestimmtheit*) that marks the final limit of all quantitative change – the maximum or minimum to which a quantum can be increased or decreased.

In Hegel's view, however, the idea of the infinitely big or small arises through combining two concepts that are explicitly incompatible. The infinitely big or small is meant to be a *quantum*, which by definition is limited and finite; and yet, as infinite, it is meant to surpass and so lie *beyond* every finite quantum and so, ultimately, *not* to be a quantum at all (SL 192, 201-2 / LS 244, 256). The idea of the infinitely big or small is thus fundamentally contradictory. As such, however, it necessarily generates a quantitative infinite progress (that differs subtly from the one described in the paragraph before last). The infinitely big or small lies beyond every finite quantum, but it is itself a quantum and so is necessarily finite; it must, therefore, lie beyond the quantum that *it* is, and it must do so endlessly, to infinity.

Yet such an endless progress beyond any given quantum is incompatible with the idea of an absolute, *unsurpassable* limit. This means in turn that there can

in fact be no such thing as the biggest or smallest quantum. Quanta can therefore be increased or decreased without limit in either direction.[6] (As noted at the end of the last chapter, this is not to deny that quantitative change is limited by the measure of things, or by the nature of material objects. Yet quantitative change as such, without regard to such a measure or nature, is in principle endless: it brings itself to no maximum or minimum.)

Hegel insists, by the way, that the quantitative infinite progress just described should not be understood as the endless process of *approaching* the infinite, absolute quantum (SL 192 / LS 244). Such an understanding would retain the idea of the infinitely big or small, but take it to be forever just out of reach. In Hegel's view, however, there can be no infinitely big or infinitely small quantum at all, for the latter "is itself implicitly [*an sich*] the infinite progress" (SL 201 / LS 256). Quanta do not, therefore, approach a maximum or minimum which sets a final limit to their change, but they continue beyond themselves endlessly. The idea that there is an absolute determinacy that could be reached, or even just approached, by quanta is thus not a well-founded philosophical idea; rather "the infinitely great and infinitely small are [. . .] representation's images [*Bilder der Vorstellungen*] which on closer inspection prove to be nebulous shadowy nullities [*nichtiger Nebel und Schatten*]" (SL 202 / LS 256). As we shall see later, this has important implications for Hegel's understanding of differential calculus: for it means that he rejects the idea of an "infinitesimal" (see 2: 214).

THE TRUE INFINITY OF THE QUANTUM

In the sphere of quality the infinite progress is not the final form of infinity: it is not true infinity. The latter, however, is implicit in that progress. This is also the case, Hegel claims, in the sphere of quantity. To discern the character of true quantitative infinity, therefore, we merely have to examine more closely the quantitative infinite progress (as it first appeared above).

That progress is generated because the quantum, as here conceived, is external to itself. This quantum has its own determinacy – at least in part – outside it in another quantum (and so comes into a further form of its determinacy by changing into that other quantum).[7] Since its determinacy lies in part outside it, the quantum continues beyond its own limit, and so relates "infinitely" to itself, in the other that it becomes. It comes to be infinitely self-relating, however, *outside* and *beyond* itself in that other. The quantum proves to be self-external, therefore, because both its determinacy *and* its infinite self-relation lie beyond it. Such self-externality, however, is what gives rise to the quantitative infinite progress: for the quantum has its infinity beyond itself in another quantum, and so does every other quantum *without end*.

In this infinite progress, Hegel claims, the quantum shows itself to be fully and explicitly what it is. The quantum is in fact characterized from the start by

self-externality, or, as Hegel puts it, "it is precisely *itself* through its externality": it contains a plurality of self-external ones within itself, and it is also one among many quanta that fall outside it but are just like it (see 2: 9-11). It is thoroughly self-external, however, in the quantitative infinite progress; accordingly, "in the infinite progress the *concept* of the quantum is *posited*" or rendered explicit (SL 202 / LS 257).

Yet this endless progress is not the last word on the quantitative infinite, because it harbours within it *true* quantitative infinity.[8] To understand why, we first need to note that in the progress the quantum is negated or "sublated". The quantum, as we know, is external to itself and continues beyond itself; yet it is also a purely self-relating one, indifferent to all other quanta (see e.g. SL 170 / LS 215). In the infinite progress, this indifferent *self*-relation is negated because the quantum is related inextricably to the infinity lying *beyond* it. Yet that is not all that is negated in the progress. What is usually overlooked, in Hegel's view, is the fact that that "beyond" also suffers negation. In the infinite progress, Hegel writes, "*there is present the sublating of the quantum, but equally of its beyond [Jenseits], therefore the negation of the quantum as well as the negation of this negation*" (SL 202 / LS 257). The quantum's "beyond" is negated because it is *not* simply beyond the quantum after all, but where the quantum relates infinitely to *itself* and its *own* determinacy.

Note that Hegel is not simply repeating an earlier point here. As we know, the quantum relates to itself and its own determinacy in another quantum lying beyond it. Indeed, it is precisely this moment of self-relation, or continuity, beyond its own limit that constitutes the *infinity* of the quantum (see SL 190 / LS 241). The fact that the quantum attains its infinity only in *another* quantum, however, places that infinity *beyond* the quantum itself, thereby generating the quantitative infinite progress. In 1.2.2.C.c Hegel now highlights a new idea: namely, that the self-relation of the quantum in the other beyond it entails the *explicit negation* of the idea of the beyond. What lies beyond the quantum is, after all, not simply beyond it but is the quantum's *own* determinacy; beyond itself, therefore, the quantum relates only to *itself* (albeit in a new form). This means, however, that the quantum, as now conceived, no longer has a "beyond" as such.

The quantum's self-relation is, of course, its infinity: its infinite relation to, and continuity with, itself in another. This infinity, however, is now understood – with a subtle shift of emphasis – to consist, not in relating to oneself *beyond* oneself, but in relating to *oneself* beyond oneself and so not actually having a beyond. Such infinity is thus itself *nothing beyond the quantum*, but rather the self-relation that the quantum *itself* proves to be. As such, it is no longer a bad infinity, but the true infinity of the quantum. This true infinity is the same relation-to-self as that bad infinity. Now, though, it does not lie beyond the quantum, but coincides with the quantum in what we might call its expanded

form: it consists in the latter's being *itself*, and being wholly *for itself*, in relating to another quantum. Being oneself and being for oneself are, however, forms of quality. The true infinity of the quantum is thus the *quality of self-relation* – the quality of being-for-self – exhibited by the quantum in its relation to another. In Hegel's own words, "the infinite, which in the infinite progress only has the empty meaning of a non-being, of an unattained but sought beyond, is in fact nothing other than *quality*" (SL 203 / LS 258).

In the quantitative infinite progress the infinity of the quantum is understood to lie beyond it in another quantum, and then beyond that quantum, too, and so on. Representation (*Vorstellung*) pictures such infinity as an infinitely big or infinitely small quantum that marks the absolute limit of all quantitative increase and decrease. The true infinity of the quantum, however, does not lie beyond the latter in another *quantum* (let alone one that is infinitely big or small). It consists in a certain *quality* that is exhibited by the quantum itself: the quality of being infinitely *self-relating* in and through another. In the last chapter, we saw quality re-emerge for the first time in the sphere of quantity, when the quantum that is both extensive and intensive turns out to belong to the *qualitative* something. This something, however, is the indifferent substrate of quanta and so does not enter into the ongoing dialectic of quantity as such (2: 153). The quality that has now emerged is different: it does not underlie, and is not indifferent to, quanta, but it is the self-relation exhibited *by* the quantum itself.

In the sphere of quality true infinity takes the form of being-for-self and, further, of the one (*Eins*). Indeed, being *one* is the most developed form of quality: being infinitely oneself. The one, however, repels or excludes itself and so is necessarily one of many. Furthermore, in being external to itself the one gives rise to a new form of being: quantity. Now we see that quantity in turn, in the shape of the quantum, reintroduces qualitative self-relation in the form of *being-for-self* (rather than merely that of being "something").[9] Moreover, it does so at the point at which it is most properly quantitative, most properly self-external and, we might say, self-alienated: it is in having both aspects of its being – its determinacy and its infinite self-relation – *outside* and *beyond* itself that the quantum relates infinitely to *itself*. It is important to note that the quantum cannot become explicitly *qualitative* before this point. The quality of being-for-self exhibited by the quantum is its true infinity; this infinity arises, however, only when the quantum is no longer simply a *finite* unit of quantity.

The quantum as such is, of course, infinitely self-relating from the start insofar as it is a one. Yet what makes it a quantum in particular is the fact that it is a determinate, limited and so *finite* one. Number, extensive magnitude and intensive magnitude are further forms of such finitude. With the changing of one quantum into another quantum, however, the quantum ceases being merely finite, because it *continues* to be itself in another finite quantum. At that point, therefore, the character of the quantum alters significantly: the quantum is no

longer an infinite one that is explicitly *finite*, but it becomes a finite one that is explicitly self-relating and *infinite*. This infinity then generates the quantitative infinite progress, because it lies beyond the quantum, and so is the "*non-being*" of the quantum (SL 191 / LS 243), but is inseparable from the presence of another *quantum*.

The *true* infinity of the quantum is still the "non-being" of the quantum; yet it is such non-being because it is not a quantum, or quantitative, at all, but a *quality* of being. This qualitative infinity that is not itself a quantum is not, however, anything beyond the quantum itself, but is the quality of self-relation exhibited *by* the quantum. It is the quality exhibited by the finite quantum that relates infinitely to itself in another finite quantum – a quality of infinite self-relation that could not have emerged before this point in the dialectic of quantity.[10]

THE TRULY INFINITE QUANTUM AND THE DIRECT RATIO

Note that the quantum first proves to be truly infinite in the quantitative infinite progress. Hegel now proceeds to examine the truly infinite quantum for itself and to render its logical structure more explicit. As we shall see, it turns out to have a structure very different from any we have previously encountered.

There are two things in particular to be noted about the truly infinite, qualitative quantum in its explicit form. First, it must exhibit a *fixed* determinacy or what Hegel calls "being-determinate-for-itself" (SL 203 / LS 258). The truly infinite quantum is purely self-relating or *for itself* and thereby recovers the logical structure of the pure one (*Eins*) before the latter dispersed itself into many ones and gave rise to quantity. Yet this quantum does not simply return to being one, because it remains a quantum and as such is *determinate*. Since, however, it is truly infinite it must be determinate in being purely *for itself*. The finite quantum, as we have seen, has a determinacy that continues into another quantum and so takes on a new form; that is, it changes. Insofar as the quantum is truly infinite, however, its determinacy is inseparable from the fact that the quantum is purely self-relating, purely *itself*, and does not become anything other than itself. Consequently, that determinacy cannot itself become anything other than itself and so cannot change. It must, therefore, be the unchanging, fixed determinacy that distinguishes *this* quantum in its infinite self-relation from any other: it must be the infinite quantum's "absolute determinacy". The *quality* exhibited by the truly infinite quantum is thus not only "being-for-self" (*Fürsichsein*), but also being absolutely determinate or "being-determinate-for-itself" (*Fürsichbestimmtsein*). Accordingly, Hegel writes: "the quantum no longer has infinity, being-determinate-for-itself, outside it but in it itself [*an ihm selbst*]" (SL 203 / LS 258).

Note the subtle difference between being determinate *for itself* and *in itself*. Every number has the characteristic of "being-determinate-in-itself" (*An-sich-Bestimmtsein*), since it contains its defining determinacy and limit within itself as its specific amount. This characteristic is what renders numbers quite external and indifferent to one another and so allows them to be added to and subtracted from one another (SL 170 / LS 215). Yet it is also what makes a number an extensive magnitude, whose dialectic leads to the idea that quanta and numbers are subject to change. Being determinate *an sich* thus turns out ultimately to mean not having a fixed determinacy but having one that increases and decreases. By contrast, being determinate *für sich* means, precisely, having a fixed, absolute determinacy.[11]

The second point to be noted about the truly infinite quantum is that, paradoxically, it does not exhibit the quality of self-relation – is not infinitely and absolutely itself – purely by itself. It exhibits this quality only in relation to another, and indeed as that relation. This relation, however, is no longer one of pure externality. As we saw earlier, a quantum changes into another quantum because its determinacy continues beyond or outside it; the two quanta are thus external to one another (see 2: 154-5). As truly infinite, by contrast, a quantum relates explicitly to *itself* in another quantum. The mutual externality of the two quanta concerned thereby falls within the self-relation of one of them, within the unity formed by that quantum with itself. In Hegel's own words, "the externality, which seemed to be a beyond, is determined as the quantum's *own moment*" (SL 203 / LS 259). This does not mean that such externality now disappears altogether. The two quanta, as quanta, remain external to one another; but they are not just that, because they are also inseparable *moments* of the unity formed by the first of them in its infinite relating-to-itself-in-the-other. "The quantum is thus posited as repelled from itself, and with that there are two quanta which are, however, sublated, only moments of *one unity*".

As we know, insofar as a quantum is finite, it changes into another quantum that lies outside it; insofar as a quantum is explicitly *infinite*, however, it is unchanging with a fixed determinacy. The two quanta it encompasses (one of which is itself) are, therefore, in that respect not moments of a changing identity, one of which is what the other becomes, but moments of a single fixed identity. As such, they are simply two quanta in relation to one another – a relation fixed by the identity and determinacy to which they belong. The unity of the infinite quantum with itself thus coincides with its relation to another quantum. That is to say, the quantum exhibits the *quality* of self-relation only in and as a *quantitative* relation to another: in Hegel's words, the quantum, "posited as qualitative, is the *quantitative relation* [*quantitatives Verhältnis*]" (SL 203 / LS 259). As just indicated, this means that the fixed, "absolute" determinacy of the infinite quantum must itself determine that relation – an idea that plays an important role in what is to come.

Hegel summarizes the logical structure of the truly infinite quantum in the final lines of 1.2.2.C.c:

> In the relation [*Verhältnis*], the quantum is external to itself, is distinguished from itself; this its externality is the relation [*Beziehung*] of one quantum to another, each of which counts [*gilt*] only in its relation to its other; and this relation [*Beziehung*] constitutes the determinacy of the quantum, which as such is a unity. In this unity the quantum has not an indifferent but a qualitative determination; in this its externality it has returned into itself and, in it, is what it is.
>
> —SL 204 / LS 259

This structure is clearly complex, but there is a further complexity to be noted before we can proceed. The point at issue is implicit in 1.2.2.C.c, but it becomes explicit later in the text, when Hegel again takes up the main thread of the logical development after the remarks on differential calculus. The point is that we need to distinguish carefully between the quantum *in* relation to another and *as* that relation. The logical structure of the infinite quantum entails both, but the difference between them is important.

Insofar as the infinite quantum stands *in* relation to another, it is one side or *moment* of that relation. As such, it is inseparably bound to the other and lacks independence. It is thus not indifferent to the other – like the cardinal number – and does not just have its determinacy within itself; rather, it has that determinacy in, and in part through, its *relation* to the other and so in part through the *other* to which it relates. Furthermore, since the infinite quantum is one moment of the relation, the other must, of course, also be a moment. This means that it, too, has its identity only in its relation to the first. Thus, Hegel writes, "the two are not related to each other as external quanta, but *each has its determinacy in this relation to the other*. [. . .] [W]hat each is, it is in the other; the other constitutes the determinacy of each" (SL 271 / LS 350).

Note carefully what it means here to have one's determinacy in and through another. An intensive magnitude has its determinacy through another insofar as it is wholly determined *by* other such magnitudes to be the degree it is. A concrete quantum that is both intensive and extensive is then no longer simply determined by other quanta, but its determinacy continues beyond it *in* the other into which it changes. Insofar as it *continues* beyond itself in another, however, a quantum relates "infinitely" to itself. A truly infinite quantum finds itself in another, therefore, not simply by changing into another and not through being wholly determined by others, but by continuing to be *itself* – and so being "for itself" – in another (see 2: 164-5). In its explicit form, however, such an infinitely self-relating quantum has to be understood both as the relation between itself and the other (in which it relates to itself), and as one *moment* of

that relation; and insofar as it is merely such a moment, as just noted, it owes its determinacy in part to the other to which it relates. As a moment of a relation, therefore, the infinite quantum finds itself and its determinacy in another in a way that is reminiscent of the intensive magnitude: for it is partly determined *by* (and reciprocally determines) its counterpart. Yet, insofar as each quantum is determined by the *other*, each remains one-sided and in that sense finite. The infinite quantum *in* relation to another – that is, as a moment – is thus not fully and explicitly infinite after all.

Hegel points out, however, that "the quantum is not only *in relation* [*im Verhältnis*], but *it is itself posited as relation* [*als Verhältnis*]" (SL 271 / LS 350). As we have just seen, the infinite quantum remains in one respect one-sided: it is *this* quantum that relates to itself in, and so is partly determined by, *another* quantum. As *infinite*, however, it relates to *itself* in relating to the other: in the other it is in unity with itself. As such, it is not just this quantum in relation to another, but one wholly *self-relating* quantum. This self-relation in turn incorporates and coincides with the relation between itself and its other. Accordingly, the infinite quantum in this respect is no longer one-sided and finite but a "self-enclosed totality" *as relation*.

There is thus a fundamental ambiguity in the structure of the truly infinite quantum in its explicit form: it stands *in* relation to another quantum, but it also contains within itself, and so coincides with, that whole relation itself. In coinciding with the relation between two quanta (one of which is itself as a one-sided moment), the infinite quantum is, however, also a wholly *self-relating unity*, and so an independent quantum in its own right. As such, it has its own "absolute determinacy" (SL 203 / LS 259). The latter, therefore, must be the determinacy of that quantum itself *and* of the relation between the two quanta: it must define and fix that relation. The intimate connection between the unity of the infinite quantum, its determinacy and the relation between the quanta is expressed, in a somewhat condensed form, in a line from the passage cited above. As Hegel notes, there are now two quanta, "each of which counts only in its relation to its other; and this *relation* constitutes the *determinacy* of the quantum, which as such is a *unity*" (SL 204 / LS 259, emphasis added). When a quantum is conceived as infinitely or "absolutely" determinate for itself and as thereby determining the relation between two quanta (one of which is itself), it is conceived as the *exponent* of that relation (SL 272 / LS 351). The quanta in what Hegel calls a "quantitative relation" or "ratio" are thus doubly determined. Each has its determinacy in and through the other, but each is further, and ultimately, determined by the exponent that governs their relation.

The truly infinite quantum (in its explicit form) has a complex logical structure, even by Hegel's standards: it is *one* self-relating quantum that coincides with, and – as exponent – determines, the *relation* between two quanta: itself and its other.[12] Note that the *same* quantum is both one side and

the exponent of the relation. As the exponent, this quantum is truly infinite and self-contained, and so is a fixed, unchangeable determinacy. As such, it fixes the relation between the two quanta. As one side of the relation, however, the quantum is both infinite and finite; that is to say, it shares the fixed determinacy of the exponent, but at the same time is a *variable* quantum, like all other finite quanta. Accordingly, the relation between the quanta is itself both fixed and variable at the same time. This sounds absurd, but is in fact quite intelligible. The two quanta in the relation change, like other quanta, but in so doing they preserve a fixed relation to one another that is governed by the exponent.

This logical structure is made necessary by the development that precedes it, but it manifests itself in a relation between quanta with which all of us are familiar. This relation is the direct ratio (*das direkte Verhältnis*). We will examine this ratio in more detail in chapter 9 of this volume, but it is worth giving a brief sketch of it now in order to show how it embodies the structure of true quantitative infinity.

Take two numbers, 1 and 4. These are two changeable quanta that can be increased or decreased without end.[13] Assume, however, that the number 4 is also the exponent of the ratio. As such, it is fixed and determines the relation between 1 and 4, understood as changeable numbers. Now assume that 1 changes to 2. As this happens, the exponent requires the relation between the two numbers to remain invariant. The change of 1 to 2 thus requires 4 to become 8, so that the ratio between them is preserved. Whether the two numbers concerned are 1 and 4, or 2 and 8, or 3 and 12, and so on, the ratio between them, determined by the exponent 4, always remains 1 : 4.

In this example we can see the logical structure of the truly infinite quantum embodied (if not in every respect) (see 2: 181). Two changeable numbers stand in relation to one another. One of them, however, is also the fixed exponent of, and so determines, their relation; it thus determines the way one of the two must change, if the other changes. A change duly occurs in one of the numbers: 1 changes to 2. The change in the other number, 4, is then determined by this change of 1 to 2, because the relation between them is in turn determined and fixed by the exponent: as 1 becomes 2, 4 must become 8, so that the ratio between them, 1 : 4, is preserved. In this way, the direct ratio is not just a relation between finite numbers, but it manifests the true infinity and "absolute determinacy" of the quantum. Such infinity is thus to be found, not beyond all finite quanta in a quantum that is infinitely big or small, but in a simple quantitative ratio with which we are all familiar. As always with Hegel, the true infinite does not lie beyond the here and now, but is present before our very eyes.[14]

CHAPTER EIGHT

Excursus: Hegel and Kant's First Antinomy

KANT'S FIRST ANTINOMY: THE THESIS, ITS PROOF AND HEGEL'S CRITIQUE

Between his accounts of the quantitative infinite progress and the true infinity of the quantum Hegel inserts two remarks, the second of which examines Kant's first antinomy. Hegel's complaint against the first antinomy is the same as that directed earlier in the *Logic* against the second, namely that the thesis, antithesis and their proofs simply *assert* two moments of quantity in opposition to one another (albeit in the guise of two opposed assertions about the "world"). In the second antinomy, Hegel contends, these moments are those in quantity as such, namely discreteness and continuity. In the first, by contrast, they are the moments of the "quantitative limit": the fact that "*there is a limit*" and that "*the limit must be transcended*" (SL 198 / LS 251-2). Hegel recognizes that in both antinomies Kant's proofs of the thesis and antithesis are meant to be apagogic. He argues, however, that in the first as in the second these proofs in fact *presuppose* what they are supposed to prove.[1]

The thesis of the first antinomy states that "the world has a beginning in time, and in space it is also enclosed within limits [*in Grenzen*]" (CPR B 454). In Hegel's view, space and time are "examples of pure quantity", so this thesis in fact contains the claim that quantity has a limit, a point at which it begins and / or ends.[2] Kant's proof of the thesis is divided into two parts, one concerning time and the other space. Since, however, the proof concerning space "also relies on time", we will look only at its first part (SL 198 / LS 252). This proceeds as follows:

1. Assume that the world has no beginning in time.
2. If this is the case, then up to every given point in time an eternity has already elapsed.
3. That is to say, there has already "passed away" (*verflossen*) an infinite series of successive states of things.
4. An *infinite* series, however, can never be completed. (In Kant's words, it can be never be completed "through a successive synthesis".)
5. Accordingly, it is impossible for an infinite world-series already to have elapsed (for it would then have been completed).
6. Therefore, the series of past events preceding a given moment in time cannot be infinite.
7. Such a series must, therefore, be finite and the world must have a beginning in time (CPR B 454).[3]

In his remark on the first antinomy Kant carefully distinguishes this proof of the thesis from another he could have given (see CPR 458-60). This alternative proof would proceed as follows:

1. A magnitude is "infinite" if no greater magnitude is possible, that is, if it is the greatest magnitude there can be or the "*maximum*".
2. Such a maximum is, however, impossible, since "one or more units can always be added to it".
3. Therefore, an *infinite* given magnitude, and so an *infinite* world (including an infinite series of past states or events), is impossible, and the world must have a limit or beginning in time (and a limit in space).

Kant rejects this "proof" of the thesis because it does not invoke what he takes to be the true concept of infinity, but simply exposes the incoherence in the "defective" (*fehlerhaft*) idea that an infinite magnitude is a "maximum" (CPR B 458-60). For Kant, as for Hegel, there is no quantitative maximum, or greatest number, beyond which one cannot go, because one can always add a unit to such a "maximum" and so go beyond it (see SL 192 / LS 243-4).[4] Quantitative infinity properly conceived, in Kant's view, is rather that which itself lies *beyond* any given magnitude. It is thus defined not by how great it is, but by the *relation* in which it stands to any arbitrarily assumed unit or number: the relation of being "greater than any number". For Hegel, of course, the quantitative infinite that lies "beyond" any given quantum gives rise to the quantitative infinite progress, which is a form of "bad" infinity.[5] A similar connection is discerned by Kant: for if the quantitative infinite is "greater than any number", then the process of traversing a series of numbers or units in order to reach that infinite is endless. For Kant, however, such endlessness is the mark, not of "bad" infinity, but of *true* infinity. In his words, therefore, "the true (transcendental) concept

of infinity is that the successive synthesis of unity in the traversal of a quantum can never be completed" (CPR B 460).[6]

Kant's proof of the thesis in the first antinomy does not, therefore, merely expose as incoherent the misconception that the quantitative infinite is a "maximum", but it shows – or purports to show – that a truly infinite world-series, which is coherent in itself, cannot be completed and so cannot be understood to have elapsed at a given point in time.[7] The temporal series of states and events that has elapsed at that point must thus be finite, and there must be a beginning to the world in time, as the thesis states.

Now, for Kant, no antinomies are generated by space and time as such, as pure forms of intuition. Antinomies are generated only when a spatio-temporal totality or "world" is held to be given and it is assumed that such a *world* must be either finite or infinite.[8] In the first antinomy the opposition is, more specifically, between the claim that this world has a beginning or limit in time and space and the claim that it does not have one. For Hegel, however, Kant's antinomy simply pits the two moments of the quantitative limit against one another: its both being and not being a definite limit. Moreover, as noted above, Hegel understands space and time as such to be examples of pure quantity. In his view, therefore, "the opposition [*Gegensatz*] could just as well be considered with respect to time and space themselves", even though he is aware that, for *Kant*, the antinomy concerns the limitedness or unlimitedness of "*the world in time and space*" (SL 198 / LS 251).

As we saw on 1: 313-14, Kant insists that, aside from the illegitimate assumption that a "world" exists, the "proofs of the fourfold antinomy are not semblances [*Blendwerke*] but well grounded" (CPR B 535). In Hegel's view, by contrast, they simply presuppose what they are meant to prove. The thesis of the first antinomy, for example, is said by Hegel to be presupposed in the argument with which the proof of that thesis begins.

This thesis states that the world has a beginning in time, and this is equated by Hegel with the claim that time has a definite *limit*: if, as it were, we scroll time backwards, we will not be able to do so indefinitely, but at some point we will stop, namely at the point at which time began. Kant's proof of this thesis is apagogic, since it starts from the idea that the world does *not* have a beginning in time and, by showing this idea to be unsustainable, leads to the conclusion that the thesis must be true. The immediate consequence of the idea that the world has *no* beginning, Kant tells us, is that "up to every given point in time an eternity has elapsed" (a consequence that is in fact an impossibility) (CPR B 454). In Hegel's view, however, such a "*given point in time* has no other meaning than that of a determinate *limit* in time". As we have just noted, the thesis, for him, is itself just the claim that time has a definite limit. In the proof, therefore, as Hegel sees it, "a limit to time is *presupposed* as actual; but *that* is just what was *to be proved*" (SL 199 / LS 252-3).

At first sight, however, Hegel's claim seems unpersuasive, for, as he himself acknowledges, the given point of time, or "now", that is assumed in Kant's proof is "the end of the previously elapsed time", whereas the "now" that is to be *proven* is "the beginning of a future", namely the beginning of the future that is now for us the past (SL 199 / LS 253). It is not clear, therefore, that the proof begins by presupposing the same point of, or limit to, time as that asserted in the thesis. Hegel's critique of the proof appears, rather, to rest on conflating in a cavalier manner two limits to time that even he understands to be quite different.

Yet Hegel is not in fact being cavalier. His claim is, rather, that by conceiving of the "now" as the definitive end of an infinite series in time, Kant turns it *into* a beginning that is indistinguishable from the beginning affirmed by the thesis.

In Hegel's view, if a given point in time, or "now", were conceived as what it in fact is, namely a *quantitative* limit that leads beyond itself to another "now" that leads beyond itself and so on, then the infinite time series that precedes this "now" would not *elapse* in the latter but would "go on flowing". In this case, however, such an infinite series would not fall into contradiction, because it would not be an *endless* series that comes to an *end* in the "now". There would be no reason, therefore, to conclude that an infinite time series cannot precede the "now", and so the proof of the thesis that such a series must have a beginning, and be finite, would collapse (SL 199 / LS 253). The idea of an infinite time series proves to be contradictory only if the "now" is conceived as its end, that is, as its *qualitative* limit. Such a limit, however, interrupts the continuity of time, and that in turn turns it into the *beginning* of a new series of moments. The idea that an infinite temporal series comes to an end does not, therefore, just prove through its internal contradiction that time must have a beginning, but it is *itself* already the idea that time has a beginning.

Yet is there not a difference between this new beginning and the beginning of time, or of the world in time, posited in the thesis? Surely, the new beginning to which Hegel refers presupposes a past that has ended, whereas the beginning posited in the thesis is an absolute beginning. In Hegel's view, however, the qualitative limit (or "now") that completes (or is thought to complete) an infinite temporal series "*breaks off*" that portion of time and leaves it "without connection" to the future, which thus has its own "absolute beginning". The idea that a temporal series has now elapsed just *is*, therefore, the idea that time, or the world in it, has an absolute beginning. If someone still wishes to insist that there is a difference between the supposed first "absolute beginning" and any subsequent one, we need only point out that *any* absolute beginning to time can be preceded by a past from which it is disconnected. So the beginning to time posited in the thesis is in fact no different from any other beginning made necessary by the completion and end of a preceding time series: each is the same absolute beginning of the time that is to come. The thesis states simply that "the world has a beginning in time", and nothing further is said about

whether this beginning is or is not preceded by any elapsed time; the claim is simply that at some point there is a *beginning*. This thesis as it stands, however, is immediately assumed to be true as soon as one thinks that an infinite time- or world-series comes to a definitive *end*.[9]

Note that Hegel does not think that any "now" is in fact a qualitative limit within time, so he does not endorse the thesis as its stands.[10] His claim, however, is that Kant's proof of the thesis itself presupposes that thesis. The proof proceeds by first assuming that the world has no beginning in time and that an infinite time has thus *now* elapsed, and then arguing that, logically, an infinite, endless "world-series" cannot in fact have elapsed since it would then have been completed and come to an end. According to Hegel, however, the idea that the world has a beginning in time is not actually *proven* at all, since it is already built into the initial assumption that an infinite time series has been completed: for where there is a qualitative end to the continuity of time, there must then also be a beginning. The proof of the thesis is thus just the assertion of it: the one-sided assertion that "*there is a limit*" to or within time and thus also quantity.

KANT'S FIRST ANTINOMY: THE ANTITHESIS, ITS PROOF AND HEGEL'S CRITIQUE

The antithesis of Kant's first antinomy states that "the world has no beginning and no bounds [*keine Grenzen*] in space, but is infinite with regard to both time and space" (CPR B 455). The proof of the time part of this antithesis proceeds as follows:

1. Assume that the world has a beginning in time.
2. This beginning must be an "existence" (*Dasein*) preceded by a time in which "the world was not"; that is, it must be preceded by an "empty time".
3. In an empty time, however, no "arising" (*Entstehen*) of anything is possible, because no part of such a time can contain, more than another, "any distinguishing condition" of a thing's existence rather than non-existence. (There can be no difference between the parts of an empty time that would explain why something should arise *now*, rather than *now*.)
4. Thus, although many things can arise *in* the world, the world itself can have no beginning and so extends infinitely into the past.

As Michelle Grier points out, the core of the proof lies in claim 3: in an empty time, there could be no distinction between "conditions" and so nothing to explain why the world should begin at a particular point in time rather than another. If the world is to have a beginning, it must be preceded by an empty

time; in such a time, however, there can be nothing to ground any beginning; the world thus cannot have a beginning after all. Claim 3, by the way, is not just that *we* cannot distinguish between conditions in empty time; the emptiness of such time itself makes it impossible for it to contain the ground of the world's emergence.[11]

Hegel's objection to this proof is once again that it presupposes the very thing it is supposed to prove. The proof of the antithesis assumes at the outset that the *thesis* is true and the world has a beginning. This beginning in turn is taken to be preceded by an empty time in which the world does not yet exist. That empty time, however, is assumed to contain the condition *of* the world's emergence. It is true that Kant does not say this explicitly in the proof; but the fact that he argues that *no* such condition can be contained in an empty time indicates that, according to the thesis, one is *meant* to be found there. The thesis – that the world has a beginning in time – thus assumes that existence extends back beyond such a beginning, and so beyond its limit, to include its preceding condition. This assumption, however, is itself the antithesis that is to be proven.

Kant's proof of the antithesis seeks to undermine the thesis that the world begins in time by arguing that the empty time before that beginning cannot contain the condition of the world's emergence and so the world cannot in fact ever begin. This proof could not succeed, however, unless the thesis were to imply that the condition of the world's emergence *must* lie in the preceding empty time. Yet, according to Hegel, in this very implication, built into the thesis, the antithesis is itself assumed to be true. As Hegel puts it, "the world is an existence [*Dasein*]" and "the proof *presupposes* that this existence *arises*" – has a beginning. In so doing, it also presupposes that this "arising has an *antecedent condition* in time". This means in turn that, in presupposing the thesis, the proof "*continues the existence of the world beyond itself into this empty time*" and "consequently *extends the existence to infinity*" (SL 200 / LS 254). The claim that the world "always *demands an antecedent condition*" and so has "no absolute limit" is, however, the claim contained in the antithesis. For Hegel, therefore, the thesis that the world begins in time and so has a definitive limit – as Kant presents it – itself presupposes the antithesis that it has no such limit but extends endlessly to infinity. The antithesis is thus not first proven by the contradiction that Kant finds in the thesis, but is implicitly taken for granted in that thesis itself.

THE RESOLUTION OF THE FIRST ANTINOMY: QUANTITY, SPACE AND TIME

As we saw on 1: 336-7, Kant resolves the antinomies by denying that there is such a thing as the "world". Reason requires that we form the idea of the world and that we judge the latter to be both finite and infinite. For the critical

philosopher, however, no genuine contradiction arises thereby, since no *actual* "world", or totality of appearances, can ever be given to us, about which mutually exclusive judgements have to be made. Hegel's resolution of Kant's antinomies is different. In his view, these antinomies, though ostensibly about the "world", in fact set two opposed *categories* against one another. The resolution of the antinomies consists, therefore, in comprehending those categories as a speculative unity (in the course of speculative logic). Hegel resolves the second antinomy, for example, by conceiving of the two categories concerned – continuity and discreteness – as explicitly containing one another, and thereby constituting quantity that is *continuous* but infinitely divisible into *discrete* units.[12]

In the case of the first antinomy, Hegel's resolution involves understanding quantity both to contain real limits, and so to take the form of finite quanta, *and* to continue beyond such limits, and so to be unending and infinite. These two aspects of quantity are united most profoundly, and speculatively, in true quantitative infinity, in which one finite quantum relates infinitely to itself (and thereby exhibits the *quality* of being-for-self) in another finite quantum (see 2: 164-5). A more immediate resolution of the first antinomy is found, however, in the idea of the quantitative infinite progress, in which finite quanta continue beyond themselves and their quantitative limits by forming an endless series.[13] This resolution – which is suggested by Hegel's text, rather than explicitly stated to be one – clearly comes closer to the antithesis than to the thesis, but it differs subtly from the antithesis. According to the latter (as Hegel understands it), quantity has no limit except "a *sublated* one", and so has no beginning or end. By contrast, Hegel's resolution involves recognizing quantity to have definite *limits* (expressed, for example, by numbers), but understanding each quantitative limit to lead beyond itself to further quantity, "in which there arises, however, another such limit, which is no limit", and so on (SL 201 / LS 255-6). (This inextricable logical connection between a quantitative limit and its transcendence then serves to explain why the thesis itself presupposes the antithesis in Kant's proof of the latter.)

In resolving the second antinomy, Hegel sets to one side Kant's cosmological terminology and conceives of quantity as continuous but infinitely divisible, but he also understands space and time themselves in the same way.[14] In the case of the first antinomy, Hegel's conception of quantity as an infinite progress also underlies his conceptions of space and time, though this is not made clear until the philosophy of nature. Regarding time, for example, Hegel says that "in tracing out the context of the finite, its antecedents must be sought [*muss man dieses Vor aufsuchen*], e.g. in the history of the earth or of humanity. In this one reaches no end" (EN 16 / 27 [§ 247 A]). And regarding space, he states that "however remotely I place a star, I can go beyond it, for the universe is nowhere nailed up with boards. This is the complete externality of space" (EN 29 / 43

[§ 254 A]). There are, however, subtle differences between quantity and space as Hegel conceives them.

Quantity as such, for Hegel, has no intrinsic maximum or minimum, but can increase or decrease endlessly without limit (though things may not exceed their *measure* without ceasing to be what they are). Space, however, is not just pure quantity as such, but quantity "existing immediately and externally". More precisely, it exists as "*side-by-sideness*" or "being-next-to-one-another" (*Nebeneinander*) (EN 28 / 41 [§ 254]); that is to say, it is all there *at once*, spread out to infinity. It is not clear, therefore, that space can increase or change its magnitude in the way that simple quantity can. Things in space can grow and diminish, but space is all there together as the "everywhere" within which things grow and so, it would seem, cannot grow itself.[15] This raises a problem, however: for how can space be all there at once and yet continue endlessly? How can it be all there and thus complete and yet also be a bad infinity that is never complete? To my knowledge, Hegel does not address this problem, but a possible solution to it is offered by Einstein's conception of space as "finite yet unbounded". Space, on this conception, would still be "endless" in the sense that one could travel in a straight line forever without end. Yet the curvature of space would mean that one would eventually end up back where one started; and this in turn would mean that space could have a "finite volume" and so be there all at once.[16] Whether this conception of space would resolve the problem raised above, or is even fully intelligible, is not something I can consider here, but it is worthy of further investigation.[17]

Time, as Hegel conceives it, is also an "example" of quantity (SL 156 / LS 197), but it does not raise the same problem as space, because its moments are not all there at once but *pass away*. Pure quantity, too, takes the form of quanta that are subject to limitless *change* (if we disregard the measures of the things to which quanta belong); such change itself, however, belongs to the unchanging logical structure of quantity. Similarly, time is the endless passing away of its moments that is itself unchanging: it is *always* the passing away of its moments. In this sense, time is "eternal" (*ewig*). "Eternity", Hegel claims, "will not come to be, nor was it, but it *is*" (EN 36 / 50 [§ 258 A]). Indeed, it is just *being* as such, not as sheer indeterminacy, but as "the absolute present, the now without before or after" (EN 15 / 26 [§ 247 A]). Time *as* eternal, therefore, is itself just being as such, indeed the same being as space. Yet whereas space for itself is sheer self-externality or "*side-by-sideness*", time is being or space as "negativity", as passing away (EN 33-4 / 47-8 [§ 257]). Time does not itself pass away, but it is being, presence, space *as* the eternal process of passing away. If we focus on the moments that have gone or those that are to come, then time will be understood as a bad infinity stretching back into the past and forward into the future, and Hegel does not deny that time takes this form: as noted above, if we follow the sequence of events back into the past "one reaches no end" (just as

one reaches no end of the number series) (EN 16 / 27 [§ 247 A]). If, however, we focus on time as the constant, eternal passing away of the present, of the space that is there, then we bring to mind the *true* infinity of time and "the question of a beginning at once disappears". Time is thought neither to have a beginning, nor to lack one, but simply to be the continuous, ever present, self-negating of space and of the things in it.[18]

Quantity as such is *truly* infinite insofar as it takes the form of a quantitative ratio, rather than an endless sequence. The true infinity of time is its eternity, its irreducible ever-presence. The true infinity of space, if there is such a thing, is perhaps its self-differentiation into points, lines, planes and enclosing surfaces, that is, into the objects of geometry.[19] Time and space, however, are topics for the philosophy of nature. Our task now is to consider in more detail the ratios in which the true infinity of the quantum (in its explicit form) consists.[20]

CHAPTER NINE

The Quantitative Relation or Ratio (*Verhältnis*)

THE DIRECT RATIO

I noted above that not every aspect of the truly infinite quantum is manifested in the direct ratio (see 2: 170). What is missing is the idea that in the relation between two quanta one of them must relate explicitly to *itself* in the other. The direct ratio can, of course, hold between two instances of the same number – say, 1 : 1 – and in that sense relate a number to itself. Yet this self-relation is not intrinsic to the direct ratio, since the latter can just as easily hold between two different numbers, in which case neither relates to itself in the other. As we shall see, this deficiency is remedied in the relation, or ratio, of powers. It mars the direct ratio, however, because the latter is the truly infinite quantum in its *immediacy*.[1] It is thus the relation between one immediately given quantum and another immediately given quantum that may, but usually will not, be the same.

In other respects, however, the direct ratio does manifest the structure of the truly infinite quantum and for this reason constitutes the initial, immediate form *of* the latter. This ratio involves two finite quanta, each of which has its determinacy through the other (since the changing of one changes the other) (SL 272 / LS 351). The two are reciprocally determined, rather than just mutually external, because they are the two sides or "moments" of one relation. This relation in turn has its own determinacy, which is governed by – indeed, is that of – an explicitly infinite quantum. The latter is *infinite*, not through being unimaginably great or small, but through being purely *self*-relating: it does not change into another quantum, but is simply itself and *for itself*. Its determinacy is thus also unchanging and fixed. Through this determinacy, therefore, the infinite quantum fixes the relation between the two finite quanta and so serves

as the *exponent* of that relation. Accordingly, in the direct ratio, two quanta stand in an unchanging relation governed by a quantum that is absolutely determinate.[2]

Since this infinite quantum is *quantitative*, it is, as Hegel puts it, "*self*-relating" in its own "*externality*" (SL 272 / LS 351). As purely self-relating, however, it is not essentially connected to anything outside or beyond itself. Accordingly, it must contain the moment of externality – of otherness and difference – within itself: it must have "its own difference" (*den Unterschied seiner*) explicit within it. Now a quantum as such contains difference by virtue of the fact that it encompasses many ones. It is fully determinate, however, only when those many ones form a definite amount that makes it a specific number: all quanta are alike in being unities or units, but they differ explicitly from one another through their amounts. The principal difference inherent *within* the fully determinate quantum, therefore, is that between its *amount* (*Anzahl*) and its *unity* (*Einheit*) (see SL 169 / LS 214). This applies to the quantum that is infinite, as much as to the one that is finite. The infinite quantum, however, coincides with the relation between two finite quanta (see 2: 169). Accordingly, the logical difference, internal to the infinite quantum, between unity – or being a unit – and amount must itself coincide with the relation and difference between those quanta. As Hegel writes, therefore, "unity and amount were at first the moments of the quantum; now, in the ratio, in the quantum as realized so far, each of its moments appears as *a quantum on its own*" (SL 272 / LS 352).

The two quanta in the direct ratio are thus not just different quanta, but they are related as unity, or unit, and amount. Note that the quanta must stand in this logical relation to one another: they are the moments of the quantitative relation with which the infinite quantum, in its self-relation, *coincides* and so they must express the logical difference that belongs to the latter. If, therefore, in the direct ratio 1 : 4, 1 counts as the unit, 4 cannot just be another, indifferent quantum but must be the *amount* of such units: "the one has with respect to the other only the value of the unit, not of an amount, the other only that of the amount" (SL 273 / LS 352).

Yet the two quanta are not only related in this way, but they are also in that relation as finite *quanta*. As such, they are external and indifferent to one another after all and, moreover, changeable.[3] As finite quanta, therefore, neither 1 nor 4 is fixed in itself, but each can become another quantum and another, indefinitely. In changing, however, they must remain related as unit and amount. Furthermore, since that relation coincides with the infinite quantum, it is fixed by the latter, which is thus the exponent of the relation: whatever the value of the unit on one side of the relation, there must always be an unvarying amount of such units on the other side. Each quantum, as a finite quantum, can change without limit by increasing or decreasing itself; in so doing, however, it must

remain in a fixed relation to its counterpart. In that sense, the change in the quanta is *limited* by the relation they have to one another and by the exponent that determines that relation.

Take the ratio 1 : 4 and let 1 count as the unit. The amount of such units is thus fixed as 4, and the latter in turn is the exponent of the ratio. Now 1, as a finite quantum, is changeable, and so may change into 2 or 3, thereby altering the numerical value of the unit. If this happens, however, the relation between unit and amount must remain constant: whatever the new value of the unit may be, the amount of such units in the other quantum must still be 4. This does not mean that the other quantum has to retain the particular value 4, but it must always be equivalent to 4 *of the new unit*. So, if the first quantum changes from 1 to 2 and then to 3, the second must change in turn from 4 to 8 and then to 12. To preserve the ratio, therefore, the numerical value of the second quantum must be changed when that of the first is changed. In this way, the second quantum is determined both by the first quantum and by the exponent that governs their relation.

The key to understanding the logical structure of the direct ratio is to recognize that the two quanta concerned stand in a twofold relation to one another. On the one hand, they are both finite *quanta* and, as such, are external to one another and changeable. On the other hand, they are the two *moments* – namely unit and amount – of a single unchanging relation. This latter relation is the direct ratio itself; but it is inseparable from the relation between changeable quanta. The direct ratio is thus not merely a fixed, unchanging relation between quanta, but one that remains fixed *as those quanta themselves undergo change*. Hegel summarizes the main features of the direct ratio in the following lines:

> When the quantum of the one side is determined, the other is also determined by the exponent and it is a matter of total indifference how the first is determined; it no longer has any meaning as a determinate quantum on its own [*für sich*] but can just as well be any other quantum without altering the determinacy of the ratio, which rests solely on the exponent. The one which is taken as unit always remains unit however great it becomes, and the other, however great it too thereby becomes, must remain the *same* amount of that unit.
>
> —SL 272-3 / LS 352

The two quanta in the ratio are related as unit and amount, because the distinction between these logical moments falls within the infinite quantum and the latter coincides with the relation between the quanta. The infinite quantum is, however, also a self-relating quantum *in its own right*, and as such, indeed, is the exponent of the relation (see 2: 169-70). As this exponent, the infinite quantum also contains the moments of unit (or unity) and amount. It contains

them, however, fused into a single *unity*; or, as Hegel puts it, “it has the meaning of both determinations immediately in it” (SL 272 / LS 352).

On the one hand, the exponent is a quantum and so contains an amount. The latter determines the ratio between the two one-sided quanta, because it is itself the amount that one quantum must contain *of* the unit represented by the other quantum. The amount in the exponent of the ratio is thus the same as the amount-side of the ratio itself. “If”, Hegel writes, “the one side of the ratio which is taken as unit is expressed as a numerical one [*Eins*] – and counts only as such – then the other, the amount, is the quantum of the exponent itself” (SL 272 / LS 352).

On the other hand, the exponent is not only an amount, but also a unit or self-relating unity. In this respect its amount is itself a “simple determinacy” – the *one* fixed determinacy that governs the relation between the two quanta. Note that in the exponent the moment of unity (or being *one*) belongs to the amount itself, but is also distinct from the latter, since it is not tied to a specific amount: it requires the ratio between quanta to be invariant, *whatever* the amount of the exponent may be. Yet amount and unity essentially coincide, since the exponent determines the ratio to be invariant through the unity and self-sameness *of* its amount. Amount and unity (or unit) thus form a unity in the exponent, whereas they are distributed between two distinct quanta in the ratio itself. They have to be related in these two ways because they are the logical components of the infinite quantum, and the latter is *one* self-relating quantum that coincides with the relation between *two* quanta, one of which is a one-sided, relational manifestation of the infinite quantum itself.

THE TRANSITION TO THE INVERSE RATIO

The two quanta in the direct ratio are mutually external, changeable units of quantity; yet that is not the whole story, for they are also related to one another as *unit* and *amount*. This, Hegel maintains, prevents them from being quanta in the full sense. As we saw earlier, a quantum is fully determinate as a number, and a number, logically, is both a unit (or unity) and an amount. Each of the two quanta in the direct ratio, however, represents just one of those two moments: “the one has with respect to the other only the value of the unit, not of an amount, the other only that of the amount” (SL 273 / LS 352). In this sense, the two quanta fall short of being quanta in the full sense: from a logical point of view – or, as Hegel puts it, “*according to their conceptual determinacy* [*Begriffsbestimmtheit*]” – “they are themselves *not complete* quanta”.

The quanta in the direct ratio thus suffer *negation*: they are reduced from being mutually external, independent units to being – at least in one respect – mere one-sided moments. Of course, quanta already suffer negation insofar as they change into other quanta, for they thereby turn into what they are not. Yet

such negation does not fundamentally alter their character as quanta: they retain their unity and an (albeit changing) amount, and they remain external to one another precisely by changing into *other* quanta. In the direct ratio, by contrast, the quanta suffer more profound negation, for they cease being complete quanta in their own right and thereby lose their independence. This loss of independence consists in being merely a unit to the other's amount or an amount to the other's unit; but it is also evident in the fact that a change in the value of one is directly determined *by* a change in the value of the other (and by the exponent of the relation). In Hegel's words,

> this incompleteness is a negation in them [the quanta], and it is so not because of their general variability, according to which one of them (either of the two) can assume all possible magnitudes, but because they are so determined that, when one is altered, the other is increased or decreased by just as much [*um ebensoviel*].
>
> —SL 273 / LS 352

The degree, of course, is also determined *by* other quanta, but it is thereby given an independent identity of its own as this or that degree. It thus remains a complete quantum that is both a unit *and* an (albeit sublated) amount. Indeed, the very fact that the degree has "*its* amount" is what converts it back into an extensive magnitude (SL 185 / LS 235).[4] Moreover, the degree is not directly determined or changed by the amount *in* any specific external degree, but derives its identity simply from the amount *of* other degrees outside it. It does not stand in direct relation to another quantum, therefore, and does not just provide the unit for the other's amount, or vice versa. It is thus not an explicitly *one-sided*, incomplete quantum like the quanta in the direct ratio.

In contrast to the two quanta in the direct ratio, which are logically incomplete, the exponent of that ratio is, or is supposed to be, "the complete quantum", since it combines within itself the two moments of unity *and* amount (SL 273 / LS 353). Yet it, too, falls short of being a complete quantum, because, considered as the "quotient", it only has "the value of *amount* or of *unit*". This deficiency of the exponent in the direct ratio stems from the fact that "there is nothing present to determine which of the two sides of the ratio must be taken as the unit or as the amount".

Take two quanta, A and B, in a direct ratio to one another, and let A be the unit. B must, therefore, be a certain amount of that unit. To find that amount – to find how many such units are in B – we simply divide B by A. The result is the quotient, C. In Hegel's words, "if one side, the quantum B, is measured in terms of quantum A as unit, then the quotient C is the amount of such units" (SL 273 / LS 353). Now a unit, as such, is always *one* unit, and so always counts as 1, whatever its actual numerical value. Since the amount of units in B is C,

the ratio of unit to amount is thus 1 : C. This ratio is to be preserved in all cases: whatever the value of the unit A, B must always be the equivalent of C units. C is thus not only one of the moments of the invariant ratio, but it is also the *exponent* of the ratio. The quotient produced by dividing B by the unit, A, thus yields the exponent, which is itself an *amount*. So, let the unit, A, be 3 and B, which contains a certain amount of such units, be 12; dividing 12 by 3 yields the quotient, 4; the ratio of unit to amount is thus 1 : 4, where 4 is both one moment of the invariant ratio and also the exponent of that ratio.

The logical functions of A and B can, however, be switched around: A can be regarded as the *amount* and B as that amount of *units*. If A is 3, and B is 12, B will thus be the equivalent of 3 sets of units. The task will then be to find the numerical value of the unit itself. This task is fulfilled by once again dividing B by A and producing the quotient, C, which is again 4. In this case, however, the quotient does not contain the *amount* of units there must be in B, but the value of "the *unit*, which with the amount A is required for the quantum B" (SL 273 / LS 353, emphasis added). This quotient also serves as the exponent of the ratio, since it demands that, whatever amount of units there is, each unit must have the numerical value of 4. The invariant relation is thus not between one unit and a certain *amount* (namely 4), but between one amount and the value of the *unit* (again 4). For each amount of units, the value of each unit must always be 4: so where A, the amount of units, is 3, B must be 3 × 4, or 12.

The mathematics here is simple. What interests Hegel, however, are the logical categories that are at play; and what has become apparent is that, logically, the exponent of the direct ratio is not fully and explicitly what it is. The exponent is the immediate unity of unity (or unit) and amount: it is a definite amount that is a simple unity, or an *absolute*, invariant determinacy, and, as such, it governs the relation between two quanta.

Yet, conceived as the *quotient*, produced by dividing one of those quanta by the other, the exponent proves to be *either* an amount *or* the value of a unit. In the direct ratio, therefore, the exponent is logically contradictory.

Note that, as a quotient, the exponent coincides with the *relation* between quanta: it is equal to one quantum *divided by* the other. This is appropriate, for, as we have seen, it is inherent in the logical structure of the infinite quantum that it coincides with the relation between two quanta (one of which is itself) (see SL 203-4 / LS 259).[5] The problem with the direct ratio, however, is that the exponent does not coincide completely and explicitly with the relation between those quanta. This is because, *as* the quotient produced by the quanta, the exponent is not what it is *as exponent*: it is not the *unity* of unity and amount, but is first one and then the other.[6] The exponent, however, is one self-relating quantum and so must be one with, and not at odds with, itself. As such, therefore, the exponent implicitly points beyond the direct ratio to a new one

in which the exponent is the *unity* of unit and amount, not only in itself, but also as the *relation* between quanta. This new ratio is the *inverse ratio*.

In the inverse ratio the exponent again coincides with the relation between two quanta: A – the unit – and B – a certain amount. As that relation, however, the exponent is no longer what it was as a quotient: either the unit *or* the amount. It is now "the *unity of both moments*" (SL 273-4 / LS 353-4). This does not mean that unit A and amount B are simply added together in the exponent: for that would just yield another sum or amount. If unit and amount are to be truly united, the exponent must be the *product* of both moments: it must be a certain amount *of* the unit, or the unit taken *a certain amount of times*. The exponent will thus not just be A plus B, and certainly not B divided by A, but A *multiplied* by B. So, if the unit is 3, and the amount is 4, the exponent will not be their sum, 7, but *four 3s*, or 12.

Any change in the unit or amount will thus be limited by the fact that the product of the two quanta (one times the other), which is the exponent of their relation, must always be the same. So if the unit increases from 3 to 6, the amount must decrease from 4 to 2, so that the product remains 12. The unit and amount are, therefore, now *inversely* proportional to one another: as one increases, the other decreases, and vice versa. It is important to note, however, that the one does not simply decrease by the *amount* by which the other increases; that would just mean that they form a constant sum or amount. A ratio is properly inverse, only when the exponent is the *product* of multiplying each side by the other; so when one increases, the other must decrease by however much is required to preserve that product. Hegel fails to see this in the first edition of the *Logic* and understands the exponent of the inverse ratio to be merely the "sum [*Summe*] of both sides" (WLS 218, 221). In the second edition, however, he corrects his earlier error and replaces the idea of a "sum" with that of a "product" (see SL 274-5 / LS 354-5).[7] Note that Hegel does not revise his logic in this case in response to contingent developments in history or science; this is never legitimate. He revises it for the only reason that is legitimate: namely that he comes to a deeper understanding of the logic itself.[8]

We turn now to examine the inverse ratio in more detail. As we will see, this ratio has a double effect: for as the exponent has become explicitly what it is, explicitly *itself*, the quanta, whose relation it governs, are negated even more thoroughly than they are in the direct ratio.

THE INVERSE RATIO

In the direct ratio the exponent has the numerical value of either the unit *or* the required amount of such units. Either way, it is a quantum and thus itself a certain amount. Accordingly, it is usually, or "primarily" (*vorzugsweise*), identified with the required *amount* of units in the ratio: "one side was the

unity and was to be taken as the unit [*Eins*], with respect to which the other is a fixed amount, which is at the same time the exponent" (SL 274 / LS 354).[9] The exponent is qualitative, since it is fixed and so "for itself"; yet it is an immediately given quantum that is identical to one side of the relation it governs.

In the inverse ratio, the exponent is also an immediate quantum that is taken as fixed. It is, however, no longer identical with just one side of the relation: it is not simply a fixed *amount* of units that must be preserved when the unit changes its value. This is because the exponent is now the product of both sides of the relation – of the unit and the amount. In the inverse ratio, therefore, "if as the unit on one side another quantum is taken, the other side no longer remains the *same amount* of units of the first side" (SL 274 / LS 354). There is thus no longer an unchanging direct ratio between the two sides. There is still a direct ratio between them, but it is now changeable.

Take two quanta, A, the unit, and B, a certain amount of such units, where A is 3 and B is 12. The direct ratio between the two is thus 1 : 4. Now assume that these two quanta are in inverse ratio to one another. The exponent of that ratio will thus be the product of the two, or 36. If we now increase the value of A from 3 to 4, the value of B must decrease from 12 to 9 to preserve the value of the exponent. The direct ratio of 4 to 9, however, is 1 : 2.25, no longer 1 : 4. So not only do A and B change (as they can also do in a direct ratio), but such change alters their *direct ratio*, too. The exponent of the inverse ratio remains "an immediate quantum only arbitrarily assumed as fixed"; yet "it does not preserve itself as such in the side of the ratio, but this side, and with it the direct ratio of the sides, is alterable" (SL 274 / LS 354).

Note that the two quanta in an inverse ratio are *negated* more thoroughly than in the direct ratio. In the latter, the quanta suffer negation because they are no longer complete, independent quanta: not only is one changed by a change in the other, but each is reduced to being either a unit *or* an amount of such units. As sides of the ratio, however, they also have a fixed value that is preserved in such change. In the inverse ratio, the quanta are subjected to more profound negation, because they no longer have any fixed value at all, but are rendered completely changeable. A change in A brings about a change in B, both insofar as the latter is a simple quantum *and* insofar as it stands in a direct ratio to A; and, conversely, a change in B thoroughly alters A.

The two quanta are negated in this way, and rendered completely changeable, by the fact that the exponent of their relation consists in their unchanging *product*. Since their product remains constant, when one side increases, the other does not increase to preserve the ratio between them, but it decreases. Accordingly, both they and their direct ratio are altered. Furthermore, the two do not just increase and decrease by the same amount, but "the one becomes *as many times* [*sovielmal*] smaller as the other becomes greater" (SL 275 / LS 356,

emphasis added): so, in the inverse ratio of 3 to 12, if 3 is multiplied by 3 to become 9, 12 is divided by 3 to become 4.[10] This gives rise to a series of varying pairs of quanta, all bound together by sharing the same product.

The exponent of the inverse ratio not only negates the quanta in that ratio by depriving them of fixity, but in so doing it also distinguishes itself from them more clearly than occurs in the direct ratio. The exponent of the direct ratio differs from the two related quanta, since it is not just one side or moment of a relation but an infinitely self-relating quantum. Yet it is also identical in value to one side of the relation, and both quanta – as moments of the ratio, rather than simple quanta – share the formal fixity of the exponent. In the inverse ratio, by contrast, the exponent sets itself apart from the two quanta, whose relation it governs, in two new ways.

First, it is a wholly distinct quantum, whose numerical value differs from that of either of the related quanta – unless, of course, one of the latter has the value of 1 – since it is their product. Second, it is fixed and unchanging, whereas the two related quanta are utterly changeable. As such, it sets a definite limit to the changes that the quanta can undergo: they can change in any way they please, *provided* their product remains constant. This limit is not one to which the quanta in the relation, or the exponent itself, are indifferent (and which they can thus exceed). It is thus not a quantitative limit – a limit that is not a limit – but it is a firm, definitive, *qualitative* limit: the limit that gives the inverse ratio its distinctive character. The exponent of the inverse ratio thus sets its self-relating, fixed, *qualitative* character against the purely changeable *quantitative* character of the two quanta. In Hegel's words,

> In the ratio now before us, the exponent as the determining quantum is thus posited as negative towards itself as a quantum of the ratio, and hence as qualitative, as limit; the result is that the qualitative moment comes to the fore for itself as distinct from the quantitative moment.
>
> —SL 274 / LS 354

To recapitulate: in the inverse ratio the exponent *coincides* explicitly with the relation between the two one-sided quanta. This is because it is not only a fixed quantum in its own right, but also the product *of* the two quanta. Furthermore, in both cases the exponent is the unity of unity, or unit, and amount. As a fixed quantum it is such a unity because it is *one* invariant amount; and as a product it is also such a unity, because it is not just a simple amount but an amount multiplied by the value of a unit (with either quantum playing the role of the one or the other). Yet, despite all this, the exponent also *distinguishes* itself clearly from the quanta in the relation by setting its own qualitative character against their merely quantitative character. There is, however, no tension between these two aspects of the exponent: for it is precisely by

coinciding with the *relation* between the quanta – with their *product* – that the exponent distinguishes itself from the changeable *quanta* themselves.

In the inverse ratio, therefore, the exponent comes more explicitly into its own. An exponent in general is an infinitely self-relating quantum that determines the relation between two quanta. In the direct ratio, however, it is identical with one of the quanta and takes its determinacy from the latter. As a consequence, Hegel suggests, the exponent is not the "determinant [*das Bestimmende*] of the ratio" that it is meant to be (SL 273 / LS 353): for the very determinacy through which it determines the relation between the quanta is given *by* one of the quanta themselves.

In the inverse ratio, the exponent remains an immediate quantum, or some specific number. Yet it coincides, not with one of the quanta in the ratio, but with their product. In this way it *frees* itself – to a certain extent – from the quanta themselves. It is true that the exponent is the product *of* those quanta; but it is nonetheless distinct *from* them, and enjoys a certain independence, since it remains constant while they change. It thereby becomes explicitly what the exponent of the direct ratio was merely implicitly, namely the factor that *determines* the two quanta in the ratio. Those quanta can change in any way they choose, but their product must always be the same; the latter, therefore, sets an absolute limit to any change of the former.

In being determined by their product, the changing quanta cease being *given* quanta in relation to one another, as in the direct ratio – quanta such as 1 : 4, or 2 : 5 – and become mere fluctuating *moments of their product*. At the same time, the exponent, in setting a fixed limit to the change of the quanta, asserts its own independent character – that is, its qualitative being-for-self – against those quanta. It thus proves itself to be more explicitly qualitative, and less bound to a given quantum, than the exponent of the direct ratio. In the inverse ratio, the exponent and the two quanta, whose relation it governs, thus undergo contrasting developments. The exponent becomes more properly *itself* – namely, the unity of unit and amount – and more independent of any given quanta, while the quanta themselves become more thoroughly relational and more dependent on their unchanging product (that is, on their exponent). Yet the exponent remains the product *of* the changing quanta in the ratio, and, of course, it is a specific, given quantum itself. So the exponent is not purely self-relating and self-determining, and so not unequivocally qualitative, after all. It is thus still imperfectly infinite.

FURTHER COMPLEXITY IN THE INVERSE RATIO

In 1.2.3.B.2 Hegel examines the inverse ratio more closely and renders explicit "the entanglement of the affirmative with the negative that is contained in it" (SL 275 / LS 355). In this ratio, he first notes, there is "an immediate magnitude"

or "*present* [*seiend*], affirmative quantum". This quantum is, of course, the exponent of the ratio. Second, this exponent is not only a quantum in its own right but also the product of the two quanta in the ratio, that is, their thorough *unity*. This unity in turn has a double character.

On the one hand, Hegel writes, this unity is that through which the two quanta are what they are, or "the affirmative moment by which they are quanta". The quanta in the inverse ratio have no fixed value but, in their variability, are always moments *of* their unchanging product or unity with one another. They owe what they *are*, therefore, not just to themselves, but to that unity; it is the latter that determines the value of the one for any value of the other. On the other hand, the unity or product of the two quanta also subjects the latter to a certain *negation*. This is because it pits them against one another as two distinct, one-sided and so limited quanta, one of which decreases as the other increases.

Insofar as each quantum owes its determinacy to the product it forms with another, it is constituted as much by the other as by itself. In that sense, it is not just itself, but the unity of itself and its other. It is thus not only a moment *of* its unity with the other, but it is *itself* such a unity. Yet it is also limited with respect to, and so different from, its other. It is thus not the explicit unity of itself and its other – the product alone is that – but it is such a unity only implicitly. To put the point another way, the identity of each quantum implicitly extends beyond it to include the other quantum that determines it; or, in Hegel's words, each quantum in the inverse ratio is "limited in such a manner that it is only *implicitly identical* [*an sich identisch*] with its other" (SL 275 / LS 355) (where being "identical with" means being "one with" and "inseparable from", rather than "the same as").

To recapitulate: the exponent of the inverse ratio is, first, an affirmative quantum in its own right and, second, the unity or product of the two quanta in the ratio. This unity is in turn the "affirmative" determinacy, through which the quanta are what they are, but are also *negated* by being set against one another as limited, one-sided moments. A further point to note is that the exponent also distinguishes itself from the quanta as "the limit of their reciprocal limiting" (SL 275 / LS 355). The quanta are limited with respect to one another since they are different quanta, but they also actively determine and limit one another. This is because a change in one binds the other completely to a corresponding numerical value. Yet the degree to which one quantum, in changing, determines and limits the other, is itself *limited* by the unchanging exponent of their relation. In this latter respect, Hegel maintains, the exponent is the "negative unity" of the two quanta: it is their unity or product that stands in a *negative* relation to them by distinguishing itself from them and setting a limit to their own limiting of one another.

At this point care needs to be taken in spelling out the relation between the reciprocal limiting of the quanta and the limiting of that limiting *by the exponent*.

On the one hand, the two activities of limiting are distinct: the quanta limit one another in a myriad of different ways depending on the value of the one that changes first, whereas the exponent sets a single, uniform limit to all such reciprocal limiting. On the other hand, there is in fact no distinction between the two activities of limiting, since the reciprocal limiting of the quanta is completely determined by, and so is itself the work of, the exponent. This point is important: the quanta do not *first* limit one another and *then* find their limiting limited in turn by the exponent; rather, it is only because the two quanta form a unity or product with a fixed, limiting determinacy that they limit one another at all. As Hegel puts it, "the two moments *limit* one another within the exponent and each is the negative of the other, *since the exponent is their determinate unity*" (SL 275 / LS 355, emphasis added). This has a significant bearing on the relation between the changing quanta in the inverse ratio.

In a direct ratio, such as 1 : 4, the two quanta have values that are both fixed and independent of one another. The unit does not determine the value of the amount to be 4, but that latter value is simply given. The quanta in the inverse ratio, by contrast, have no fixed, independent values. They do, indeed, start with given values, but as one changes it determines – and limits – the value of the other. If the other then changes its new value, it determines and limits the value of the first in turn. Each limits the other, however, only because they are both moments of one fixed product, which is the exponent of the ratio. This exponent is thus what does the real work in the inverse ratio. It is what requires the quanta to determine and limit one another in the first place *and* it determines and limits how they limit one another. The quanta in an inverse ratio do not, therefore, limit one another randomly, but when each changes – starting from the value it happens to have – it reduces or increases the other *by however much it takes to preserve the product*.

The values of the quanta in the ratio are thus determined and limited both by one another and by the exponent. As one quantum increases, it decreases the other by an amount that is determined by its own increase, and in that sense it negates and limits the other. More precisely, the one decreases the other by the *factor* by which it increases: "the one becomes as many times [*sovielmal*] smaller as the other becomes greater" (SL 275 / LS 356). The first quantum is thus not simply different from and external to the second; as Hegel puts it, it "continues itself *negatively* in the other" by depriving the other of an amount that is relative to what it adds to itself.[11] Indeed, each quantum, in simply *being* itself, deprives the other of what it (the first) is: "however much it is in amount, that much it sublates in the other as amount". Each has in itself, therefore, precisely what the other does *not* have: "each has its magnitude insofar as it has in it that which [. . .] the other lacks". Accordingly, the two quanta are bound inseparably to one another, for each is the negative *of* the other. Yet they negate one another in this way, only because the *exponent* is presupposed: in depriving the other of

what it has in itself, each deprives the other of what the other needs *in order to produce the exponent*.

Recall that each quantum is "*implicitly identical* with its other", since the identity of each is determined by, and so implicitly includes, the other as much as itself (SL 275 / LS 355). Each is thus implicitly the *unity* of itself and the other. The explicit unity of the two is, of course, their product – the exponent. Logically, therefore, each quantum is implicitly (*an sich*) that product or exponent. In Hegel's words, "this unity, the whole, constitutes the *intrinsic being* [*Ansichsein*] of each, from which their *given* [*vorhanden*] magnitude is distinct" (SL 275 / LS 356).

Each quantum, however, is merely *implicitly* this unity or product, because it *lacks* in itself what would make it the latter explicitly: it lacks what would make it equivalent to the exponent. What each lacks is found in the other. Accordingly, each is deprived *by* the other of what it needs to be the exponent; or, in Hegel's words, "each *is* only to the extent that it takes from the other a part of their common intrinsic being [*Ansichsein*], the whole" (emphasis added). This sets a limit to the extent that one quantum can negate and limit the other: for, whatever its value is, it denies to the other *only* what would make the other the exponent. By observing this limit, and depriving the other of just *this* amount, each quantum ensures that both together produce that exponent. This in turn requires one to decrease by *however many times* the other increases. Both the fact that, and the way in which, each quantum negates and limits the other are thus determined by the exponent.

In the inverse ratio the two quanta do not increase or decrease together, but one goes down as the other goes up. In any given case, however, there is a limit to the decrease in one quantum that is produced by the increase in the other: as A increases, it reduces B only to the point at which A times B is equal to the value of the exponent. In Hegel's words, therefore, the change in one quantum effected by the other "has its maximum in the exponent" (SL 275 / LS 356).

Hegel maintains, however, that there is a limit or maximum, not just to the increase or decrease of the quanta in a given case, but to any possible increase of the quanta. However much the quanta change, Hegel claims, neither can become equal to the exponent, and so attain the "maximum", *by itself*: "The exponent is the *limit* of the sides of its ratio, within which limit the sides increase and decrease proportionately to each other; *but they cannot become equal to this exponent*" (SL 276 / LS 356, emphasis added). The exponent is unattainable by either side of the ratio, however, only in a *logical* sense, not insofar as it is a specific number: it is a logical limit for the quanta in the ratio, not a numerical limit. Either side can attain the numerical value of the exponent, if the other side is reduced to 1 (though Hegel does not mention this himself); but, logically, neither side alone can become the "whole" that constitutes the exponent.[12]

The problem is as follows. The exponent of the inverse ratio is an infinitely self-relating quantum; as such, it has a fixed determinacy, but it is also the product of the two quanta in the ratio. This determinacy could, of course, have been a different one, but the fact that the exponent is a *product* is logically necessary: it is built into the very idea of the inverse ratio's exponent (even though Hegel did not see this in the first edition of the *Logic*). The two quanta in the ratio must, therefore, between them yield the product that is the exponent of their ratio. As such, they must be "factors" of that product (SL 275 / LS 355) and one-sided moments of the ratio itself. Note that they can never cease being such moments: for, were they to do so, the exponent would cease being their product and so would no longer be what it is. It would cease being a truly *infinite* quantum that coincides with the relation between two quanta.

Recall that each quantum is a *moment* of the inverse ratio "insofar as it limits the other and is thereby limited by it" (SL 276 / LS 356). The quantum, however, "loses" – or, rather, would lose – "this, its determination, by making itself equal to its intrinsic being", that is, by making itself equal to the exponent. It would lose that determination for two reasons. First, if one became equal logically to the exponent, it would become equal *on its own* to the whole product of the two and so would eliminate its counterpart. The other quantum, as Hegel puts it, would be reduced to nothing, to a logical "zero" (*Null*), and this in turn would mean that the first quantum is no longer limited by anything and in that sense loses its "determination". (Note that neither quantum can be reduced *numerically* to zero, since that would make their product, and thus the exponent itself, zero. A quantum in an inverse ratio can be reduced numerically to zero only if the exponent is the sum, rather than the product, of two numbers: for in that case, if the whole sum falls in one of the quanta, there is nothing left for the other.)

Second, the quantum that becomes the whole product would itself "vanish" (SL 276 / LS 356). Once again Hegel's point is a logical one. If one of the quanta were to become equal logically to the *whole* product of the two, it would cease being merely one *moment* in relation to another. As one of the quanta in the ratio, however, its very nature is to be one moment or side. In ceasing to be a moment, therefore, it would lose its logical character or determination and in that sense would vanish. Indeed, the ratio itself would vanish, since logically there would be no moments in relation to one another. That means in turn that the exponent would cease being the *product* of such moments and no longer coincide, in its infinite self-relation, with a relation between two finite quanta; that is to say, it would cease being a truly *infinite* quantum. It is thus impossible for one of the quanta in an inverse ratio to become equal logically to the whole product that constitutes the exponent, and so the latter represents a logical "maximum" that they can never attain.

In Hegel's view, each quantum in the inverse ratio is logically contradictory: it is a finite, one-sided moment, but is implicitly "the unity of the whole" (SL 276 /

LS 356). It can, however, never become explicitly what it is implicitly without ceasing to be a one-sided moment and thereby bringing to an end the ratio itself and with it the true infinity of its exponent. Logically, therefore, the two finite quanta, as moments of the ratio, can only ever *approach* the infinite exponent: the latter is "their *beyond*, to which they *infinitely* approximate but which they cannot reach" (SL 276 / LS 356-7). As one quantum gets bigger, it approaches the value of the exponent; *logically*, however, it can never become the exponent itself, because the latter, as the product of both quanta, is beyond *either one of them*. The process of change in which the quanta are engaged, as one increases and the other decreases, is thus an "infinite progress" towards an infinite but unreachable beyond. This remains true even if one of them attains or surpasses the numerical value of exponent: for in that case the quantum concerned is still one *moment* in relation to another (whose value is 1 or a fraction) and so, from a logical point of view, does not become the "whole" all by itself. The whole – the exponent – remains the product of the two and so beyond either one taken by itself.

Note that the quantitative infinite progress that emerges here is subtly different from the one that emerged earlier. That earlier progress arises as one finite quantum changes into another *finite* quantum and so relates "infinitely" to itself *beyond* itself, and this process then continues indefinitely (see 2: 161). The progress is not, however, the process of endlessly *approaching* a distinct quantum that is infinitely big or small, since there can be no such thing; there is, therefore, only the endless progress itself (see SL 192 / LS 244).[13] The infinite progress that arises with the inverse ratio differs from this in that it *is* the process of approaching a distinct infinite quantum. That infinite quantum, however, is not infinitely big or small, but the *qualitatively* infinite – and so fixed – exponent.

Since this quantum lies beyond the quanta in the ratio (logically, if not numerically), it is a peculiar hybrid that we have not encountered before: the truly – that is, qualitatively – infinite quantum that is at the same time a *bad* infinite.[14] This bad infinite, it should be noted, differs from the original bad infinite in the sphere of quality. The latter is the immediate infinite that has *turned into* a limited, finite, and so bad, infinite but that is not explicitly limited and finite from the outset. The new bad infinite, by contrast, is from the start the *explicit* limit and negation of the quanta in the inverse ratio. Accordingly, as Hegel puts it, "the bad infinite is equally *posited* here as what it is in *truth*, namely, only the *negative moment* as such" (SL 276 / LS 357). Yet this infinite is not only the negation of the quanta – not only their logical beyond – but also an affirmative presence or "*affirmative this-sidedness*" (*affirmatives Diesseits*). It is present as "the simple quantum of the exponent", and so is a beyond that is actually "attained" (*erreicht*) in the exponent.

The exponent of the *direct* ratio is an infinitely self-relating quantum, since it has a fixed determinacy that determines the relation between two quanta. As

infinite, however, it coincides immediately with one of the finite quanta in the relation. The exponent of the inverse ratio differentiates itself from the quanta in the ratio that it governs by coinciding with their product, rather than just with one of them. In this way, it is more independent, and so more explicitly self-relating and infinite, than the exponent of the direct ratio. Yet the exponent of the inverse ratio is a decidedly ambiguous infinite. It is *truly* infinite in that it displays the quality of being-for-self. Yet it is a *bad* infinite, since, logically, it lies beyond the quanta in the ratio, and it limits the way they limit one another. Yet it is not simply a perpetual beyond, since it is *present* in the form of an actual quantum.[15]

The ambiguously infinite exponent of the inverse ratio thereby represents an ambiguous blend of quality and quantity. It is qualitative, as just noted, since it is being-for-self, and so has a fixed determinacy; and it sets a definitive limit to the quanta in the inverse ratio, to which they cannot be indifferent. Yet its qualitative fixity is tied to a specific quantum that could just as easily have been a different one.

There is also a further ambiguity to the exponent of the inverse ratio. It is more independent of the two related quanta, and so more explicitly self-relating and qualitative, than the exponent of the direct ratio; but it also fulfils more clearly what was prefigured in the idea of the *quantum*, specifically in the idea of the quantum as a number. The qualitative one (*Eins*), we recall, is both truly infinite and limited, but the limit it contains is immersed in its infinite self-relation (see e.g. SL 132 / LS 166). Accordingly, the one does not limit something that is clearly different from it, that is clearly *not* it, but it relates solely to itself – both in the form of itself and of many other *ones*. The quantum as number is also an infinitely self-relating one, but it differs from the simple one by having an *explicit* limit with respect to another such quantum. This limit, which gives the quantum a distinctive determinacy, is found in the specific amount of units it contains. The quantum as number would thus appear to be the explicit unity of true infinity and limit, more than a hundred pages before the exponent of the inverse ratio emerges. This finite, limited quantum, however, though a one and therefore infinitely self-relating, is also essentially *self-external* – a feature that leads to its having its determinacy outside itself and so having to change into another quantum to be itself. As such, the finite quantum is thus not fully self-contained or "for itself", and so is not *truly infinite*. As we have seen, the quantum becomes truly infinite – that is, truly infinite *explicitly* – only when it no longer changes endlessly into other quanta, but is a fixed quantum that coincides with the relation between two quanta – that is, when it proves to be the exponent of a quantitative ratio.[16] Furthermore, only the exponent of the *inverse* ratio combines true infinity explicitly with a clear and distinct *limit* – one that distinguishes it from the quanta in the ratio. The exponent of the inverse ratio thus has a different logical structure from

that of the quantum as number, the finite quantum. Nonetheless, insofar as it explicitly unites being infinitely self-relating and being limited, it is a further development *of* that quantum. It is, we might say, the finite quantum as it is in its truth.

Yet the inverse ratio and its exponent do not bring the story of quantity and the quantum to an end, but there is a further ratio to be considered: the ratio of powers. Hegel claims that this ratio is implicit in the inverse ratio, as the latter has now been understood.

THE TRANSITION TO THE RATIO OF POWERS

At the start of 1.2.3.B.3 Hegel summarizes the logical structure of the inverse ratio. This, he writes, consists in the fact

> that a quantum is immediate, but at the same time related to an other such that it is greater by as much [*um so viel*] as the other is smaller, that it is what it is by virtue of relating negatively to the other; also, a third magnitude is the common limitation [*Schranke*] of this their increase in magnitude.
>
> —SL 276 / LS 357

What characterizes the two quanta in the ratio, we are told, is the fact that they grow bigger or smaller; that is, they change. By contrast, the third quantum – the exponent – that limits this change is a "*fixed* limit" and, as such, exhibits the quality of being-for-self. The two quanta, as we have seen, can approach this fixed, qualitative limit, but they can never become equal to it logically: neither can ever become on its own the exponent that is the product of both of them. They are thus "*variable* magnitudes, for which that fixed limit is an infinite beyond" (logically, if not numerically). In the inverse ratio, as it is here conceived, therefore, the exponent is fundamentally *different* from the quanta whose change it regulates: it is unchanging, whereas they are variable, and logically it is different from either of them, since it is their product.[17] Hegel now goes on to show, however, that the exponent is not just different from, and does not just lie beyond, the other two quanta after all.

First, Hegel reminds us that the exponent is not a *sheer* beyond in itself, since it is "at the same time some *present* [*gegenwärtig*] finite quantum" (emphasis added) – though, of course, this by itself does not make it any less different from the other quanta. Second, Hegel maintains that the "*fixity*" (*Festigkeit*) of the exponent – through which it is qualitative, rather than purely quantitative – "has developed itself as the mediation of itself with itself in its other, the finite moments of the relation" (SL 277 / LS 358). This fact that the fixed exponent is "mediated" with, and so relates to, itself in those finite moments – in the other quanta – does undermine the difference between the latter and the former.

The exponent relates to itself in those quanta, because they are in fact its own moments. The exponent is, after all, the product *of* those quanta; it thus "includes them within itself", and, even though logically they differ from it, "in them it *implicitly* [*an sich*] relates to itself" (SL 277 / LS 358). There is, however, more to be said: for, as Hegel puts it, the exponent does not just include the quanta within it "immediately" (*unmittelbar*). This means, I take it, that it does not just include them in their immediacy as *given* quanta. The exponent has a fixed determinacy – say, 36 – which is the product of, and so contains, two quanta, A and B; yet this does not mean that it contains 3 and 12 in particular, as opposed to 9 and 4, or 6 and 6. So in what sense exactly are A and B contained in the exponent? In what precise sense does the latter relate to itself, or "mediate" itself with itself, in those quanta? The answer lies in this somewhat forbidding sentence:

> But in the inverse ratio difference has developed into the *externality* of quantitative being, and the qualitative is not merely the fixity [of the exponent], and does not just include the moments immediately within itself, but it is present as uniting *with itself* in the *otherness that is outside itself* [*außersichseienden Anderssein*].
>
> —SL 277 / LS 358

These lines are certainly not the most lucid Hegel ever wrote, but his point is not as elusive as it appears. Recall that the "externality" of quantitative being leads directly to the idea that quanta are subject to *change*: a quantum changes because its determinacy continues beyond or *outside* it in another quantum.[18] Hegel's claim in the first two lines of the above sentence, therefore, is that in the inverse ratio the difference between the two quanta is one in which they are utterly self-external, *changing* entities. This is not the case in the direct ratio. There the two quanta, A and B, are, of course, changeable as simple quanta, but as moments of the ratio itself they remain fixed and unchangeable: the quanta, 1 and 4, can change into 2 and 8, but the ratio of 1 : 4 is constant. In the inverse ratio, by contrast, A and B have no fixed value at all, but are utterly changeable. The last lines of the above sentence become intelligible if one bears in mind this intimate connection between "externality" and "change": for Hegel can now be read as saying that the "qualitative" – that is, the exponent – "unites with itself", or relates to itself, in the *changing* of the two quanta in the ratio. Those quanta are included in the exponent, therefore, not in their immediacy – as these given quanta rather than those – but as varying quanta.

The idea that the qualitatively fixed exponent relates to itself in the changing of the quanta has, I think, two sides to it – one that focuses on the exponent itself, and another that focuses on the quanta. To focus first on the exponent, Hegel's claim is this: that the latter proves its *qualitative* character by showing

itself to be independent of any *given quanta*, and it does this by remaining fixed *as* those quanta *change*. It is thus precisely in the changing of the quanta that the exponent shows itself to be wholly self-relating and autonomous. This thought, however, does not yet undermine the difference between the quanta and the exponent.

If we then focus on those quanta, Hegel's claim is this: that in changing, and so negating their own immediate value and that of their counterpart, they show themselves to be implicitly *identical* to the exponent; that is, they render explicit that they are implicitly identical to the latter. The exponent in turn ceases thereby simply to differ logically from the quanta and comes to relate to itself *in* those quanta themselves. In this respect, therefore, the difference between the quanta and the exponent is now undermined, thanks simply to the changing of the quanta.

If we now put this thought together with the previous one, we get the following result: the exponent proves to be independent of all given quanta by remaining fixed as they change, and so in this respect it is mediated with itself *by* those changing quanta; yet the quanta, in changing, show themselves to be implicitly identical to the exponent, so the latter is in fact implicitly mediated with itself *by itself*. It is, as Hegel puts it, "the mediation of itself with itself in its other" (SL 277 / LS 358).

The point we now need to consider is precisely how the two quanta in the inverse ratio prove themselves, in changing, to be implicitly identical to the exponent. We touched on this point above (2: 193), but we now need to look at it in more detail. In 1.2.3.B.3 Hegel highlights two ways in which this "implicit identity" is established. First, in the process of change the quanta lose their immediacy and become what their other makes of them. This is because each is required to decrease or increase as the other increases or decreases. In the process of change, therefore, each quantum comes to have its "value in the value of the other". In so doing, however, it proves to be, implicitly, not just itself but the *unity* of itself and its other, that is to say, the product of both. The unity of the two quanta, conceived as their *explicit product*, is the exponent. In being changed by its counterpart, therefore, each quantum shows itself to be "*implicitly* [*an sich*] the whole of the exponent" (SL 277 / LS 358) (but fails to be the latter explicitly because it lacks the factor contained in the other).

The exponent is the *affirmative* unity with which each quantum is implicitly identical and that it approaches, but logically can never reach, as it increases in relation to the other. Each quantum, however, also has what Hegel calls a "negative moment" that consists in its negating and limiting of the other. This negative moment yields the second respect in which each quantum, in changing, is implicitly identical to the exponent, for the limit that each imposes on the other is not simply its own, but also belongs to the exponent: in Hegel's words, "their limit is that of the exponent" (SL 277 / LS 358). Each quantum does,

indeed, limit the other as it changes; yet precisely because they change, neither contains a fixed "immanent limit" to which it subjects the other. The fixed limit that works through each one is that of the exponent. Furthermore, the fact that one quantum has to change by the same *factor* as the other is due solely to the fact that the exponent – which is the *product* of the two – has to remain the same. The limiting of one quantum by the other is thus implicitly the work of the exponent, rather than just the quanta themselves.

In changing, therefore, the two quanta prove to be implicitly identical with the exponent in two senses: the affirmative identity of each is implicitly the product of both, and so implicitly the exponent; and the negative moment, or limit, contained in each is in fact the limit contained in the exponent. There is, however, a third respect in which each quantum is implicitly identical to the exponent, though this is itself implicit, rather than explicit, in what Hegel writes. Hegel notes that in the process of ongoing change – which takes the form of an "infinite progress" – every value of either quantum is subjected to negation. The quanta thereby prove to be bound to *no* particular value at all, that is, to have no "fixed immediacy". Yet neither quantum is simply eliminated, but each preserves a constant identity throughout. The identity it preserves, however, consists simply in *being a moment of the exponent*. In the inverse ratio, each quantum is always such a moment; but it proves ultimately to be nothing but such a moment, as it changes and so loses every particular value. In proving to be no more than a moment *of* the exponent, however, each quantum shows itself to belong completely *to* the exponent and so, in that sense, to be implicitly one with the latter.

We have seen that in the change, to which the quanta in the inverse ratio are subject, the exponent proves its independence as a "*fixed* limit" (SL 276-7 / LS 357) and distinguishes its qualitative self-relation from the quanta themselves. In this respect, the exponent relates to itself *through* the changing quanta. At the same time, however, those changing quanta prove to be implicitly identical to the exponent. In this respect, the exponent implicitly relates to itself *in* those quanta, and the logical difference between them and the exponent, which is their product, is thereby undermined: the exponent proves to be *one with the changing quanta*, at least implicitly. This, I take it, is what Hegel has in mind when he says that in the changing of the quanta (which is in principle endless), and the accompanying "negation of every particular value", we also encounter the "*negation* of the self-externality [*Außersichsein*] of the exponent" (SL 277 / LS 358). The exponent is external to itself in the inverse ratio, insofar as it lies *outside* and *beyond* the moments whose product it is, that is, beyond its *own* moments. In the changing of the quanta in the ratio, however, the exponent proves not simply to be beyond those moments after all, but implicitly to be one with them.

When this unity, or identity, of the exponent and its moments is rendered *explicit*, there arises a new quantitative ratio. In this new ratio the self-relating

of the exponent coincides completely with the changing of quanta or their "going-beyond-oneself" (*Hinausgehen über sich*) (SL 277 / LS 359).[19] Conversely, such change just *is* the self-relating of the exponent. Moreover, the exponent is now wholly independent and self-determining, since it is no longer the product of quanta that are logically different from it. As such, it must determine completely the process of change with which it coincides: the exponent of the new ratio must be the *sole* "determinant of such going-beyond-oneself". Note, too, that in this new ratio there can no longer be *three* different quanta: the exponent and two others that stand in relation. If the self-relating of the exponent coincides completely with the changing of quanta, then the quantum that relates solely to itself can be nothing beyond or apart from that which changes: it must itself *be* that which changes. Moreover, it must change into a new quantum precisely *by* relating solely to itself. Indeed, it is this quality of self-relation, rather than the quantum that exhibits it, that is the real exponent of the new ratio. What governs this ratio is not one specific quantum, but the fact that a quantum must relate to *itself* in becoming another, whatever its magnitude might be. As Hegel will show, a quantum does so by changing into its own product, that is, by becoming a new form, or higher "power", of itself. The new ratio that is implicit in, and so made necessary by, the inverse ratio is thus the ratio of powers.

Before we proceed to examine this latter ratio in detail, however, let us briefly review what has been said about the other two ratios. In doing so, there is a risk of numbing the minds of readers through repetition; yet, due to the complexity of Hegel's argument, there is a risk of missing the wood for the trees if we do not provide such a review.

The truly infinite quantum in its explicit form, we recall, is one that is wholly self-relating and so exhibits the quality of being-for-self, but that also coincides with the quantitative *relation* between quanta. In its immediacy, this infinite quantum is the exponent of a direct ratio between quanta. This exponent is an infinitely self-relating, and so fixed, quantum; but as a *quantum* it also contains a specific amount which gives it its determinacy or numerical value. It coincides with the relation between quanta, since its own determinacy is the determinacy – the defining character – *of* that relation. The relation itself is between two quanta, one of which is the unit and the other the amount. The unit-quantum is usually taken to be 1, and the amount-quantum some other number, say, 4, in which case, however the value of the unit may change, there will be always be 4 of them (so if the unit has the value of 3, the resulting amount will be 12). On the other hand, the amount can be taken to be 1, and the value of the unit to be 4, in which case, however many units there are, the value of each one will always be 4 (so if there are 3 units, the total value of them will be 12). Either way, the ratio between the quanta is fixed as 1 : 4, as opposed to 1 : 3 or 1 : 5, so it is the number 4 that gives the ratio its distinctive character or "determinacy".

This number is thus the exponent of the ratio. The exponent of the direct ratio coincides with the relation *between* quanta, therefore, by coinciding with just *one* of them.

In the inverse ratio, the exponent is still an amount, but it is no longer identified with just one of the two quanta; rather, it now coincides explicitly with their *relation* as such, since it is their *product*. Yet, as the limit that remains fixed while the quanta themselves change, the exponent sets itself apart from the latter as qualitative – as being-for-self – rather than merely quantitative and changeable. In the inverse ratio, therefore, the *quality* of the exponent comes more clearly to the fore than is the case in the direct ratio: the qualitative fixity of the exponent is no longer identical with one of the quanta, but is asserted *against* those quanta, as both different from them and as their limit.

At the same time, however, the exponent of the inverse ratio, in its qualitative self-relation, is implicitly *identical* with the changing quanta in the ratio. As we have seen, the exponent is the unity or product of the two quanta that each one is implicitly (but that, logically, neither can ever become explicitly); and the limiting of each quantum by the other is implicitly the work of the exponent itself. This implicit identity of the exponent with the changing quanta becomes explicit in the ratio of powers. In the latter, the exponent is now fully qualitative and self-relating, since it has nothing outside itself; and it has nothing outside itself because it coincides completely with the changing of one quantum into another. Yet this complete coincidence means that the quantum that is infinite and self-relating must itself be the quantum that undergoes change. It relates solely to *itself* in such change, however, only if the quantum into which it changes is itself *once again*. The quantum must therefore become a higher power of itself. In this way, the unity of qualitative self-relation and quantitative change that is implicit in the inverse ratio is rendered explicit in the ratio of powers.

THE RATIO OF POWERS

In the ratio of powers the change of one quantum into another coincides with that quantum's pure self-relation. This is to be understood in two related ways. First, the quantum remains "identical with itself" in going beyond itself and becoming another quantum, because it relates once again to *itself* in the other that it becomes. In changing, therefore, the quantum exhibits the *quality* of being-for-self (SL 278 / LS 359).[20] Second, it is by virtue of relating to itself that it changes in the first place. The quantum thus determines the manner of its change by itself. How, precisely, does this happen?

The fully determinate quantum (or number), we recall, has two logical moments: it is a unity, or unit, with a specific amount. When it changes, it then becomes a new quantum with a different amount. In the ratio of powers, however, a quantum changes through itself and by relating solely to itself. Since

its "self" is in turn distinguished by the *amount* contained in its unity, the new quantum that it becomes must thus be determined solely *by* that amount. Accordingly, it must simply be *that* amount *of* that amount. If the original amount of the quantum is 3, the amount of the quantum into which it changes will be 3 *multiplied by itself*, or 3 raised to the second power.

Note that the amount of the new quantum is the *product* of a unit (containing a certain amount) and an amount, and in this respect has a similar logical structure to the exponent of the inverse ratio. In the latter, however, the unit's own amount and the amount by which it is multiplied are external to, and thus usually different from, one another: the exponent 36, for example, is the product of 3 and 12, or of 4 and 9, and so on. (It can also be the product of 6 and 6, but this is not a necessary feature of the exponent.) In the ratio of powers, by contrast, the amount in the unit (or unity) and the amount by which the latter is multiplied are necessarily the same. As Hegel puts it, "the unit, which in its own self is amount, is at the same time the amount in relation to itself [*gegen sich*] as unit" (SL 278 / LS 359). The product of unit and amount, or the "power" to which the quantum is raised, is thus an amount or "plurality [*Menge*] of units, each of which is this plurality itself". That power is the product of the amount in the unit *and itself*.

There are two further differences between this power and the exponent of the inverse ratio. First, the latter is a quantum that remains fixed as its factors change and so is indifferent to that change, whereas the power is itself the result *of* change: it is a new quantum that arises when another is multiplied by itself. Second, and more importantly, this power, as a distinct quantum, is not itself the exponent of the ratio of powers. When the number 3 changes into its second power, 9, that change is not governed by the idea that the new number must be 9 in particular; that number is simply the by-product of multiplying 3 by itself. This is not to deny that the change of the quantum in this ratio is limited and so has an exponent. The latter, however, is not the amount of the power (in this case 9). Nor, indeed, is it the amount of the original quantum: for it is in fact not an *amount* at all.

In the direct ratio 1 : 4, the amount that is the exponent – in this case, 4 – must be preserved as the two quanta increase; so as 1 becomes 2 and then 3, 4 in turn becomes 8 and then 12, thereby preserving the ratio 1 : 4. In the inverse ratio, the exponent is the product of the two quanta, rather than being equal to one of them; but it is still a specific amount, and this amount must be preserved as the quanta change their value. In the raising of a quantum to its second power, however, there is no amount that has to be preserved or attained. When 3 is multiplied by itself, the amount, 3, is not preserved, since the result is 9. Furthermore, as just noted, the change in the first quantum is not governed by the need for it to become 9 in particular; so 9 is no more the exponent of the ratio than is 3.

So what is the exponent of the ratio of powers? What determines the change of one quantum into another? It is the fact that the quantum must change by relating to and being multiplied by *itself*. It does not matter what the initial quantum is, or what the amount of its second power is. What matters, and what governs the process of change, is simply that the quantum must raise itself to its second power or "square". The quantum, therefore, does not have to have this specific amount rather than that, but has to exhibit the *quality* of self-relation. It is this quality that is thus the exponent of the ratio of powers (or, to put it another way, the exponent is the quantum insofar as it manifests this quality). As Burbidge puts it, "the 'exponent' of this ratio is no longer an immediate quantum, but is rather qualitative – a simple requirement that the quantum be related directly to itself".[21]

In the ratio of powers, therefore, the quantum undergoing change becomes fully and explicitly qualitative and self-relating. This is because the quality it exhibits is indifferent to its amount and so does not require it to have one specific amount: a quantum can raise itself to its second power, *whatever* its amount may be, whereas the exponent of the inverse ratio has to be the specific, fixed amount that is the product of its factors. As wholly qualitative and self-relating, of course, the quantum raised to its second power is fully and explicitly *infinite*. A truly infinite quantum is thus not one that is infinitely big or small, or just one that is fixed, but one that has been squared.[22] Yet this quantum, *as* a quantum, must still have some amount or other. Furthermore, in being explicitly qualitative, this quantum is also fully *quantitative* by virtue of the fact that it is posited as "continuing in its otherness", that is, as changing (SL 278 / 359). In the ratio of powers, therefore, being purely self-relating, and so qualitative, coincides perfectly with being external to oneself, and so quantitative. The quantum that raises itself to its second power is the absolute fusion of quality and quantity: the quantum that is absolutely for itself beyond itself in the quantum into which it changes.[23]

Note that the quantum can also raise itself to further powers, and so become A^3 or A^4 and so on. It is not, however, necessary for the quantum to do this in order to become explicitly self-relating and qualitative; raising itself to the second power is sufficient. Furthermore, from a logical point of view, nothing new is introduced when a quantum raises itself to the third or fourth power. A quantum that does this simply continues or "repeats" the act of self-multiplication involved in squaring, and so reinforces the qualitative character that it already acquires through such squaring. Nothing in Hegel's account of the ratio of powers, therefore, prevents a quantum from raising itself to the third, fourth and on to the nth power; but such an endless series of powers is not a *logical* necessity.[24]

Hegel remarks that "the ratio of powers appears at first to be an external alteration to which a given quantum is subjected"; and, indeed, any finite quantum can be squared (SL 278 / LS 360). He goes on to point out, however,

that squaring is not simply an arbitrary activity imposed on quanta, but in fact renders explicit what is implicit in the quantum all along. In that sense, the quantum, and with it quantity as a whole, reaches its logical fulfilment in the ratio of powers. Quantity not only makes the squaring of numbers necessary, but it displays its nature most clearly precisely in such squaring: "in the determinate being into which it has developed in the ratio of powers, the quantum has reached its concept [*Begriff*] and has realized it most completely". The reason why this should be is easy to see.

The defining characteristic of quantity, or what Hegel calls "the quality of the quantum", is being-outside-oneself, or self-externality (SL 279 / LS 360).[25] Quantity is, of course, distinct from quality; but for that very reason it has a distinctive quality of its own: to continue outside itself, rather than to be simply itself. A "concrete" quantum is thus external to itself – and so (as we saw in 1.2.2.B.c) has its determinacy in another quantum – because that is its quality.[26] What happens in the ratio of powers is that the quantum continues outside itself in another quantum, not just because that is its quality *as a quantum*, but because it is now explicitly *qualitative* and self-relating as well as quantitative. The quantum thus proves to be "qualitative" in a more developed sense than we have seen so far: it no longer just *has* the quality that belongs to it as a quantum, but it *is* explicitly qualitative in being a quantum. In the ratio of powers, therefore, the quantum is no longer a simple quantum as such. Nonetheless, the quantum merely becomes fully and explicitly what it always is: that which continues beyond itself by virtue of its *quality*. This, I think, is what Hegel has in mind in these lines, which conclude 1.2.3.C.2:

> The *externality* of the determinacy is the quality of the quantum, and this externality is thus now posited in conformity with the concept of quantum, as the latter's own determining, *as* its relation to itself, its *quality*.
>
> —SL 279 / LS 360[27]

There is, therefore, nothing arbitrary in the fact that quanta can be not only added to and multiplied by one another but also *raised to a higher power*. Such squaring is, rather, made necessary by the nature of quantity itself. It is made necessary because in squaring itself the quantum becomes fully and explicitly what it is: namely, that which continues beyond itself. A concrete quantum continues beyond itself by having its determinacy in *another*, but the continuity of the quantum is thereby submerged in its self-externality. That moment of continuity becomes fully explicit, however, in the quantum that raises itself to a higher power: for the latter relates solely to *itself* and its *own* product in the other that it becomes. In the ratio of powers, therefore, the quantum comes to be explicitly itself in two senses: it realizes its *own* nature or "concept", and it does so by becoming explicitly *self*-relating and qualitative.

THE TRANSITION TO MEASURE

Hegel's account of quantity is complex and detailed, and it is easy to get lost in its complexity. The development of quantity, however, has a relatively simple overall structure.

Being, in Hegel's logic, first proves to be quality, and quality in turn attains its most developed form in being-for-self, or being *one*. The one is purely self-relating being – being that is purely, and exclusively, *itself*. Yet it also sets itself outside itself as many other ones, each of which it holds at bay or "repels" from itself. The one thus does not stand alone but is necessarily one of many. Yet the one also draws together with those other ones to constitute *one* single unity, which Hegel calls the "one of attraction" (SL 141 / LS 178-9). According to Hegel, therefore, it lies in the nature of the one both to repel other ones and to be attracted to, and to form a unity with, those ones. These two moments of repulsion and attraction are then understood to presuppose, and so to *continue* in, one another. Such continuity, however, is at the same time the continuity of the *one* in the many other ones to which it relates.

When the one is understood in this way to *continue outside itself* in other ones, it is understood to constitute a new form of being that is no longer simply qualitative, that no longer consists in being oneself. Such being is quantity. Quantity is thus not purely *self-relating* being – being that is itself – but being that is *external* to itself, that continues beyond its own limit (which is thus not a definitive limit). Accordingly, quantity is the continuity of discrete ones (or units), whose very discreteness extends beyond them. Note that quantity, so conceived, is not just an unexplained given, but is made necessary by quality. More specifically, it is made necessary by the fact that quality takes the form of being *one* or being a unit. Quantity is simply the being that arises because the one, or unit, continues beyond or outside itself in further ones (see 1: 293, 296).

Now in the course of its logical development, as we have seen, quantity divides itself into units that are not only discrete but also determinate and limited, that is, into quanta. The quantum in turn is fully determinate as a number that contains a definite amount of units within itself (SL 168-9 / LS 213-14). Yet the quantum continues to be external to itself. The further self-externality of the quantum (after extensive magnitude) takes two distinct forms.

First, the quantum is external to itself insofar as it is an intensive magnitude or degree. Each degree has its own determinacy and so is itself the 3rd or 4th degree; it is "self-external", however, because it is determined to be what it is by other degrees *outside* it. Every degree thus owes its determinacy to the position it occupies in a scale of degrees (SL 183-4 / LS 232-4). Second, the quantum is external to itself in a more developed sense; indeed, it is fully self-external. In this case, its determinacy is no longer simply determined *by* other quanta, but it continues beyond itself *in* other quanta. The quantum must,

therefore, *change into* another quantum in order to be itself (SL 189-90 / LS 239-40). Quantity as such is being that is external to itself and continues beyond its own limit. It comes to be fully and explicitly self-external, however, in the quantum that is required to change.

As is clear from the above, the logical development of quantity is governed principally by the idea of self-externality. This idea requires quantity to be the continuity of discrete units, and it requires quanta to change into other quanta. There is, however, an ambiguity in the idea and thus in quantity itself. On the one hand, quantity is being that continues *outside* and *beyond* itself; on the other hand, however, it is being that *continues* beyond itself. For much of its logical development the idea of being *outside* oneself predominates. Yet such self-externality is also – more or less explicitly – being that *continues*, and so relates *to itself*, in what lies outside it.

This moment of continuity and *self*-relation becomes fully explicit in the ratio of powers, in which a quantum goes beyond itself and becomes another quantum by becoming a higher power *of itself*. In that ratio the quantum in its self-externality thus exhibits the *quality* of pure being-for-self, and the logical development of quantity is brought to its completion.[28] Quantity is first made necessary by quality in the form of being-for-self and the one: it is the continuity of the one or the unit, outside itself in other units. In the ratio of powers, however, quantity returns to being purely for itself, and to being qualitative, *in* being quantitative. As Burbidge points out, therefore, we have here an example of a "double transition": from quality to quantity and then from quantity back to quality. This is not the first such transition we have encountered in the logic of being. Indeed, Hegel states, they are "of great importance for the whole of scientific method" (SL 279 / LS 361).[29]

Note that the ratio of powers does not just return us to pure quality (which would simply lead once again to quantity and so give rise to an endless cycle). It confronts us, rather, with the immediate identity of being-outside-oneself and being-for-self, or of quantity and quality: a quantum, in becoming another *quantum*, thereby constitutes explicitly *qualitative* being. This identity of quantity and quality, however, implicitly contains a further category that opens a new sphere of being.

The quantum in the ratio of powers relates to itself in relating to, and becoming, *another* quantum. As qualitative, however, the quantum is a *self-relating* quantum. Implicit in the thought of this quantum, therefore, is the following further thought: that if the quantum is to be truly *qualitative* in being quantitative, it must become wholly self-relating or *for itself*. This in turn means that it must cease – for the moment, at least – being related to another quantum. It can thus no longer raise itself to a higher power, but must relate solely to *itself* – be simply itself with "its own determinacy" – and *thereby* constitute quality (SL 288 / LS 371).[30] As such, the quantum turns, logically, into what

Hegel calls a *measure* (*Maß*). A measure is – initially – a simple, self-relating quantum with a given numerical value; but it is not merely a quantum, because it establishes, through itself, the presence of *quality*. Such quality in turn is initially immediate and so is the quality of *something*. A quantum-as-measure thus gives something, or allows it to have, the quality that makes it what it is.

Such a something, as we know, is indifferent to its quantum or magnitude as such: it can alter its magnitude – become bigger or smaller – without ceasing to be what it is.[31] A thing's measure, by contrast, is the magnitude to which it cannot be indifferent. This is because that measure is the magnitude (or range of magnitudes) through which the thing has the quality that defines it. The sphere of measure is thus a third sphere of being that is distinct from quality and quantity by being the fusion of both.[32]

CHAPTER TEN

Excursus: Hegel on Differential Calculus

Before we leave the sphere of quantity and move on to measure, we must first examine Hegel's complex account (and defence) of differential calculus. This account is set out principally in two long remarks inserted into the text of the *Logic* before the section on the quantitative relation (1.2.3).[1] The remarks contain a wealth of detail about calculus, its history and its significance, but I shall focus on what I take to be Hegel's main ideas. Hegel's *Logic* was published several decades before Karl Weierstrass provided what Carl Boyer calls "the rigorous formulation" of the calculus, so one might think that Hegel's thoughts on the latter can be at most of historical interest.[2] Yet Hegel was familiar with the work of many of the most important figures in the development of calculus towards its "rigorous formulation", such as Leonhard Euler, Joseph-Louis Lagrange and possibly Augustin-Louis Cauchy, and he had a subtle and perceptive understanding of calculus that, I would argue, should still be taken seriously today. His thoughts on the subject have been ignored by all but a few specialists, but I hope to show that they are insightful and deserve to be better known.[3]

THE PHILOSOPHICAL JUSTIFICATION OF CALCULUS

Hegel notes that calculus has produced "brilliant" (*glänzend*) results, including results that could not have been attained in other ways (SL 204-5 / LS 259-61). Yet, from his perspective in the early nineteenth century, its procedure has still not been properly justified by mathematicians. They regard that procedure as

justified simply by the fact that it works and yields correct results; but they do not justify it philosophically, or "through the concept", and so do not have a clear understanding of what is and is not legitimate in it. This in turn means that they are unable to determine the proper range of application of calculus and to protect it from misuse (SL 204 / LS 259-60). Hegel accepts that calculus can be used in a variety of ways.[4] He contends, however, that its most appropriate purpose becomes clear only when philosophy has clarified its principal concepts and determined its legitimate procedure.

In Hegel's view, the main problem with the procedure employed in calculus, as it is usually conceived, lies in its misconception of what he calls the "mathematical infinite". More specifically, it lies in conceiving of that infinite in terms of the infinitely small or "infinitesimal". This, Hegel claims, exposes calculus to criticisms to which it need not and should not be exposed (SL 204, 215 / LS 260, 276).

As we have seen, Hegel considers the idea of the infinitely big or small to be incoherent, since it arises through combining two concepts that are explicitly incompatible. The infinitely big or small is meant to be a *quantum*; and yet, as infinite, it is meant to lie *beyond* every quantum (and so, ultimately, not to be a quantum at all). It is thus fundamentally contradictory.[5] Hegel contends, however, that the mathematical infinite as it appears in calculus – as the "differential coefficient", $\frac{dy}{dx}$ – is not to be conceived in terms of the infinitely small. It rests, rather, on the "true infinite", as the latter is conceived at the end of the section on quantity (SL 204, 207-8 / LS 260, 264-5). We will see in more detail later what this means for our understanding of the mathematical infinite. What is important here is simply to note that, for Hegel, if $\frac{dy}{dx}$ is conceived on the basis of the true (quantitative) infinite, rather than the infinitesimal, then calculus is no longer exposed to criticisms directed at the latter. Furthermore, as we shall see (2: 212-13), another apparent problem with calculus also disappears. Calculus is thus shown, not only to produce brilliant results, but to be a consistent and philosophically well-founded branch of mathematics.

Hegel's aim in his remarks on calculus is thus to defend and justify calculus on the basis of the "concept", more precisely the concept of the true (quantitative) infinite. This "conceptual" justification of calculus will not alter the basic technique of differentiation: before and after Hegel, if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ (SL 235 / LS 301).[6] Hegel's account of calculus will, however, transform our understanding of the expression "$\frac{dy}{dx}$" and of the procedure through which $\frac{dy}{dx}$ is *proven* to be nx^{n-1} (in general and in given cases).

Note that Hegel's philosophical justification of calculus does not take the logic of quantity forward or deepen our understanding of the true (quantitative) infinite. For this reason, his account of calculus is not set out in the main text of the *Logic*, but is consigned to (albeit extensive) remarks. Hegel's account demonstrates, however, that the philosophical concept of true infinity is not

only of philosophical interest, but also provides a powerful tool for clarifying and defending a central branch of mathematics.

PROBLEMS WITH THE COMMON CONCEPTION OF CALCULUS

Hegel points out that dx is usually understood to represent an "*increment*, *growth*, increase of x" (SL 219 / LS 281).[7] If, however, the magnitude of y is dependent on, or a "function" of, the magnitude of x – for example, if $y = x^n$ – then an increase of x by dx must produce a corresponding increase of y by dy.[8] Accordingly, $\frac{dy}{dx}$ is to be understood as the ratio between the increase of x and that of y, whatever the specific magnitudes of dy and dx may be. It does not express the direct ratio between x and y themselves, but the ratio between the *growth* of one and the corresponding *growth* of the other. That ratio, however, will remain the same, however small dx and dy are taken to be. It can, and indeed must, therefore, be understood as the ratio between dy and dx as "*infinitely small magnitudes*" (SL 219 / LS 281). In other words, the differential coefficient, or "mathematical infinite", $\frac{dy}{dx}$, is, on the usual understanding, a ratio between infinitesimals.

How then – assuming this conception of dy and dx – are we to discover what that ratio, $\frac{dy}{dx}$, is? We start with the idea that y is a function of a variable magnitude, as in $y = x^n$. We then add dy to y and dx to x and thereby, as Hegel puts it, give the function "the form of a binomial by the addition of an increment [*Zuwachs*]" (SL 235 / LS 302): so $y = x^n$ becomes $y + dy = (x + dx)^n$. This binomial equation is then expanded in order to discover the function of x that is equal to $\frac{dy}{dx}$. Since, however, dy and dx are taken to be infinitely small, any higher powers of dx that arise in that expansion can be "left out of account" (*außer acht gelassen*) (SL 205 / LS 261-2). The value of $\frac{dy}{dx}$ will thus reside in the terms that remain once the higher powers of dx have been removed.

So, for example, if we take the equation, $y = x^3$, and add in the increments, we have $y + dy = (x + dx)^3$. If we then expand this, we get

$$y + dy = x^3 + 3x^2.dx + 3x.(dx)^2 + (dx)^3$$

Since dx is taken to be infinitely small, we can set aside its higher powers, namely $3x.(dx)^2$ and $(dx)^3$, which leaves us with

$$y + dy = x^3 + 3x^2.dx$$

The initial equation, before the increments were added, is, however, $y = x^3$, so we can eliminate y from the left side and x^3 from the right side of the equation. This leaves us with

$$dy = 3x^2.dx$$

If we then divide through by dx, we obtain the result we were looking for:

$$\frac{dy}{dx} = 3x^2$$

If we take a different equation, $y = x^4$, and add in the increments, we have $y + dy = (x + dx)^4$. If we then expand this, we get

$$y + dy = x^4 + 4x^3.dx + 6x^2.(dx)^2 + 4x.(dx)^3 + (dx)^4$$

If we then discount the higher powers of dx and so on, as in the previous case, we get

$$\frac{dy}{dx} = 4x^3$$

In both cases, therefore, if $y = x^n$, $\frac{dy}{dx}$ proves to be nx^{n-1}. Hegel does not dispute this general result, but affirms it as central to the technique of differentiating; and he goes on to claim that one can learn this technique (and the reverse technique of integration) "in a short time, in half an hour perhaps" (SL 235 / LS 301). He is critical, however, of the procedure we have just outlined for finding $\frac{dy}{dx}$, or rather of *the way the procedure is conceived*: for, so he claims, it introduces "the appearance of *inexactitude*" (*Schein der* Ungenauigkeit) into calculus (SL 205 / LS 261).

This appearance is generated by the fact that the finite magnitudes, y and x, are increased by infinitely small ones, dy and dx, which in the subsequent operation are "partly retained and partly ignored [*vernachlässigt*]" (SL 205 / LS 261). In the procedure dy and dx as such are retained, but the higher powers of dx are ignored as being too small to make a difference. Yet ignoring these higher powers makes it look as though $\frac{dy}{dx}$ does not equal nx^{n-1} exactly but only approximates to the latter (because it is actually equal to nx^{n-1} *plus* the terms of the expansion that have been ignored, or rather those terms divided by dx). This appearance of inexactitude, however, is at odds with the aim of mathematics to provide exact proofs and exact results. As Hegel puts it, "proof is essential to mathematical cognition, since it is scientific cognition"; accordingly, "in what is to be understood by mathematical determinacy any distinction between greater or lesser exactitude falls away completely" (SL 205-6 / LS 262). The procedure through which $\frac{dy}{dx}$ is found, as that procedure is usually conceived, thus conflicts with the claim of mathematics to be an exact science and thereby exposes mathematics to criticism.

Hegel insists, however, that the result reached through this procedure is not actually inexact at all: if $y = x^n$, then $\frac{dy}{dx}$ is precisely nx^{n-1}. The procedure

thus only makes it *seem* as though the result is inexact, when in fact it is not. In Hegel's words, "what is remarkable about this procedure is that, despite the admitted inexactitude, the result obtained is not merely *fairly* close, or *close enough* that the difference can *be disregarded*, but is *perfectly exact*" (SL 205 / LS 261). Yet this still leaves us with a problem: for an exact result is reached through a procedure that cannot ground that exactness but, on the contrary, makes the result look approximate (see SL 220 / LS 282).

As we shall see later, Hegel accepts that the value of $\frac{dy}{dx}$ can be found by expanding or "developing" the binomial $(x + dx)^n$.[9] Yet, by construing such expansion in what he takes to be the appropriate way, he seeks to remove the appearance of inexactitude from its result and thereby to show calculus to be a perfectly precise science. How does he do this? By arguing that "the differential of x^n is wholly exhausted [*erschöpft*] by the first term of the series which results from the expansion of $(x + dx)^n$", and so is exactly expressed by that term *alone* (SL 226 / LS 290). No "inexactitude" is introduced, therefore, by "ignoring" the subsequent terms, because they do not belong to $\frac{dy}{dx}$ in the first place. (Note that by the "first term" of the expansion Hegel understands, not x^3 or x^4 which, in our examples above, are eliminated from the right side of the equation when y is removed from the left side, but the first term after that, divided by dx, namely $3x^2$ or $4x^3$. The latter are the first terms that belong specifically to the expansion, rather than to the original function.)

Hegel further contends that $\frac{dy}{dx}$ is a *relation* or *ratio* (*Verhältnis*) completely determined "*in the first term*", rather than a *sum* (*Summe*) comprising all the terms of the expansion, some of which, however, are ignored (SL 227 / LS 290). If one conceives of the differential coefficient as the sum of all the terms of the expansion, then, clearly, omitting some of them on the grounds that they are so small as to be negligible will make it look as though the term that is left does not express that coefficient exactly (even though it does in fact do so). If, however, one conceives of $\frac{dy}{dx}$ as a relation, then (so Hegel claims) it can be expressed completely and exactly by the first term of the expansion alone; ignoring all further terms does not, therefore, introduce any appearance of "inexactitude" into the differential coefficient. Hegel thus does not deny that, in order to find $\frac{dy}{dx}$, one must set aside all but the first term of the expanded binomial $(x + dx)^n$; yet in doing so, in his view, one does not render inexact the "sum" that $\frac{dy}{dx}$ is supposed to be, but one picks out the term that exactly expresses the *relation* that $\frac{dy}{dx}$ is.

What I have just said, however, only tells part of Hegel's story. He rejects the idea that $\frac{dy}{dx}$ is a *sum* of terms, all but one of which are to be neglected because they are infinitely small. Yet, as I noted above, he also rejects as contradictory the very idea of the infinitely small. For Hegel, then, whatever is represented by a term that includes a higher power of dx, it is *not* an infinitely small magnitude, because there can be no such thing. Equally, neither dy nor dx

taken alone is to be understood as an infinitesimal magnitude. The usual conception of the procedure through which we find $\frac{dy}{dx}$ thus not only introduces the appearance of inexactitude into calculus, but it does so on the basis of the idea of an infinitesimal (and therefore in some cases negligible) magnitude that Hegel regards as a fiction or "nebulous shadowy nullity" (SL 202 / LS 256).

Yet if dy and dx are not infinitesimal magnitudes, then $\frac{dy}{dx}$ cannot be conceived as a relation between such magnitudes. So between what items is it a relation? This is what we will now consider – though to do so we first have to look more closely at Hegel's conception of the true quantitative infinite.

THE TRUE QUANTITATIVE INFINITE AND THE MATHEMATICAL INFINITE

The idea of a quantum that is infinitely small or "infinitesimal" is, in Hegel's view, a contradictory fiction: for a quantum as such can always be further diminished (or increased) and so can never be infinitely – that is, *unsurpassably* – small (or big). If, however, we take the idea of the infinitesimal seriously, and think it through, it gives rise to a quantitative infinite progress.[10] The infinitely small (or big) is meant to lie beyond any limited, finite quantum, and yet it is also meant to be a quantum itself (which by definition is limited and finite); it must, therefore, lie beyond the quantum that it is itself meant to be, and it must do so endlessly, *to infinity*. Such an endless progress beyond any given quantum is, however, incompatible with the idea of an absolute, *unsurpassable* limit; so this means, once again, that there can be no such thing as the smallest (or biggest) quantum.

Yet Hegel contends that the contradictory idea of the "infinitely great or small" implicitly points to the *true* quantitative infinite.[11] It does so, because the "infinitely great or small" properly understood is, insofar as it is *infinite*, precisely that which "can no longer be increased or diminished" and so logically is "in fact *no longer a quantum* as such" (SL 206 / LS 263). The infinitely small is meant to be infinite *as a quantum*, to be a quantum that cannot be surpassed. In truth, however, "the quantum, insofar as it is infinite, requires being thought as sublated, as something which is not a quantum". An infinite quantum, so conceived, does not cease altogether to be quantitative: in Hegel's words, it retains its "*quantitative determinacy*". Yet its *infinity* cannot consist in its being a *quantum*, whether very small or very large, but must consist, rather, in the *quality* of infinite self-relation or "being-for-self" that is exhibited by the quantum.[12] The quantum exhibits "true infinity" – and quality – in its most developed form when it raises itself to a power of itself and thereby relates to itself, or is "for itself", in the new quantum that it becomes.[13] In Hegel's view, therefore, the idea of the infinitesimal points logically to that of the truly infinite quantum that *squares* itself (once or repeatedly).

Hegel sets out the logical structure of the truly infinite quantum in its most developed form in 1.2.3.C, but he also sketches that structure briefly in his first remark on calculus. He notes, for example, that this quantum is "*in its own self* [*an ihm selbst*] infinite" (SL 207 / LS 264).[14] Indeed, he states that its infinity consists precisely *in* its simple self-relation: it is a quantum that has "returned into simplicity and the relation to itself" (SL 207 / LS 264-5).[15] On these pages Hegel does not remind us explicitly that this quantum relates to itself in raising itself to a power of itself – that quantum x is truly infinite in raising itself to x^2 (or higher powers). Yet he does highlight a further important feature of such an infinite quantum: namely, that in relating to itself it also comes to be a *moment of* its self-relation. As x becomes x^2, it exhibits the "infinite" quality of self-relation both in becoming x-by-x and in becoming the relation of x *to* itself-in-the-form-of-x-by-x. As x comes to relate to itself in this latter way, however, it also comes to be one moment of that self-relation, namely *x*-in-relation-to-x^2 (and x^2 in turn comes to be the other moment). In Hegel's view, therefore, x's being a *moment* belongs inextricably to its being *self-relating* and thus infinite.[16]

A truly infinite quantum is infinite, not insofar as it is a quantum, but insofar as it exhibits the quality of self-relation or "being-for-self". Similarly, Hegel argues, such a quantum is a moment *of* its infinite self-relation insofar as it is "no longer some finite quantum" but a "quantitative determinacy in *qualitative* form" (*Größebestimmtheit in* qualitativer *Form*) (SL 208 / LS 265). The explanation for this second claim is as follows. A quantum or number as such has "a completely indifferent existence" outside of any relation into which it may enter: 3 is 3 quite apart from its being added to 4 or multiplied by 5. By contrast, Hegel states, "the qualitative is what it is only in its distinction from" – and in relation to – "an other" (SL 216 / LS 277). This is not true of quality in the form of pure being-for-self, but it is true of being-for-one which is nothing but a moment in relation to another moment (see SL 128 / LS 161).[17] Furthermore, *something* (*Etwas*) has its defining quality only in being limited by, and so in *relation* to, another (and unlike the quantum it is not indifferent to such a limit). Something does, indeed, have an immediacy of its own, but it is properly determinate only *through* its limit, *through* its not-being-another: a meadow is a meadow only insofar as it is also not a wood (see SL 98-9 / LS 122-3).

Insofar as a quantum is a moment of its infinite self-relation – insofar as x relates *to* itself in the form of x^2 – it is also what it is only in relation to its counterpart. x is, indeed, a quantum or number with an identity of its own quite apart from the fact that it is squared; yet *as* a moment of its infinite self-relation, x is not an indifferent quantum but is *purely relational*: it is simply *x*-in-relation-to-x^2. As Hegel puts it, "as a moment, it is thus in essential unity with its other", and is "only as determined by this its other, i.e. it has meaning only with reference to that which stands in *relation* to it" (SL 208 / LS 265).[18]

The quantum, x, as a *moment* of its infinite, qualitative self-relation, is thus itself a "*qualitative determinacy*". The quality exhibited by the quantum as a moment is, however, different from that exhibited by the quantum as self-relating. The latter is the quality of "being-for-self", whereas the former is the quality of "being-for-one" or the very quality of being a mere moment.[19]

Note that Hegel does not deny that the truly infinite quantum is a *quantum* or number: it is 3^2, 4^2 or 5^2. His point, however, is that it is not *in* being a quantum that it is infinite. The infinitesimal is meant to be infinite *as a quantum*, namely infinitely *small*. By contrast, the truly infinite quantum is infinite only in exhibiting the *quality* of being-for-self. Similarly, that quantum is a moment of its qualitative infinity only insofar as it exhibits the *quality* of being-for-one.[20]

To repeat: the infinite quantum, as a *moment*, "has meaning only with reference to that which stands in *relation* to it". Hegel emphasizes the point by adding that "*outside this relation* it is a *nullity*" (SL 208 / LS 265). By this he does not mean that the x that is in relation to x^2 is not at all a quantum that can exist outside that relation; as a *quantum*, it does indeed have an "indifferent existence" of its own. As a *moment*, however, x is nothing outside the relation *of* which it is a moment. Unlike the simple quantum x, therefore, the qualitatively "momentary" x cannot exist on its own: it is merely a moment and nothing apart from that. In this sense it can be said that "that which is *only* in relation is not a quantum" with an "indifferent existence" outside that relation (SL 216 / LS 277).

Now, as we have seen, Hegel claims that the logical concept of the true quantitative infinite underlies the "mathematical infinite" or differential coefficient, $\frac{dy}{dx}$ (SL 208 / LS 265). There is, however, a significant difference between the two infinites. The true quantitative infinite (in its most developed form) is a quantum that contains a certain "amount" (*Anzahl*) like any other quantum. Yet it is *infinite* insofar as it raises itself to a power of itself and thereby relates to *itself* in the other quantum that it becomes (see SL 278 / LS 359). It is thus a finite quantum – 3, 4 or simply x – *as* infinitely self-relating, *as* "being-for-self", and so *as* qualitative. Accordingly, it leads logically to the concept of a "measure", which is a finite quantum that explicitly constitutes the defining quality of a thing.

By contrast, the mathematical infinite, $\frac{dy}{dx}$, is not a single quantum that raises itself to a power of itself and is therein "for itself", but it is a relation between two different items: dy and dx. Of course, the value of $\frac{dy}{dx}$ can involve powers and be $3x^2$ or $4x^3$; logically, however, $\frac{dy}{dx}$ as such is not a power-ratio, but simply a relation between different items. In this respect, therefore, the mathematical infinite is not *truly infinite* in the quantitative sense. Nonetheless, it reflects the true quantitative infinite in another respect: for dy and dx are not independent quanta but mere *moments* in relation to one another.

According to what Hegel regards as the usual conception of calculus, dy and dx are "increments" by which y and x are understood to grow. More specifically, they are infinitely small increments – infinitesimal quanta added to y and x (SL 219 / LS 281). In Hegel's view, by contrast, dy and dx are not infinitesimal magnitudes, or indeed quanta at all, but they are just moments in relation to one another. dx, therefore, is "essentially not a determination in relation to x" – not a quantum added to x – but it is merely a moment in relation "to dy" (SL 231 / LS 296).[21] dy and dx are thus, in being quantitative, merely *qualitative* determinations. Hegel summarizes his conception of the mathematical infinite as follows:

> dx, dy are no longer quanta, nor are they supposed to signify quanta, but they have meaning only in their relation [*Beziehung*], *a meaning merely as moments*. They are no longer *something* [*Etwas*] (something taken as a quantum), not finite differences; but they are also *not nothing*, not a null [*Null*] void of determination. Outside their relation they are pure nullities, but they are to be taken only as moments of the relation, as *determinations* of the differential coefficient $\frac{dx}{dy}$.
>
> —SL 215 / LS 275

The relation between dy and dx is thus one "in which the quantum has vanished and, consequently, the ratio is preserved only as a qualitative relation of quantity, and its sides are also preserved as qualitative moments of quantity" (SL 218 / LS 278).

The idea that dy and dx are "qualitative" has a further meaning that we will consider later (see 2: 222-5). Its initial meaning, however, is the one we have just elaborated: that dy and dx are not quanta with an independent identity outside their relation, but mere moments *of* that relation. In the case of dy and dx, being "qualitative" thus initially means no more than being "momentary". dy and dx are quantitative determinations (and so can be mistaken for "increments"), but they are in truth merely "*qualitative moments* of quantity" (emphasis added).[22]

In Hegel's view, this reduction of dy and dx to mere qualitative moments simply renders explicit what is implicit in the more familiar idea that they are "infinitesimal" quanta: for implicit in this latter idea is the thought that, as infinitesimal and so "infinite", *they can no longer be quanta.* In Hegel's words: "the so-called infinitesimals express the vanishing of the sides of the ratio as quanta, and [. . .] what remains is their quantitative relation purely determined in qualitative fashion" (SL 230 / LS 295). dy and dx are thus quantitative items related to one another qualitatively as *mere* moments of their relation, without retaining the independent, "indifferent" existence that characterizes quanta.

For Hegel, therefore, dy and dx are to be conceived on the basis of the true quantitative infinite. Yet they remain subtly different from the latter: for dy and dx are qualitative moments of quantity that are no longer quanta *at all*, whereas the true quantitative infinite is the truly infinite *quantum*. As we have seen, this quantum's infinity does not reside *in* its being a quantum, but lies rather in the quality of self-relation or being-for-self exhibited by the quantum. In this sense, we can say that the infinite quantum, insofar as it is infinite, is not a quantum but is qualitative. Equally, insofar as the quantum, x, is a mere moment of its "infinite" relation to itself – *x*-in-relation-to-x^2 – it is not a quantum either: for, unlike the latter, such a moment is not indifferent to, and cannot exist outside, the relation to which it belongs. It is nothing apart from that relation, and in that sense is "qualitative". Yet, having said this, the infinite quantum *is* still a quantum: it is a finite quantum, x, that relates "infinitely" to itself and in so doing is a moment of its self-relation. By contrast, dy and dx are mere moments and *not quanta at all*.

As purely relational moments, or "being-for-one", dy and dx thus lack "being-for-self" in any sense. They essentially relate to one another and so lack the being-for-self that belongs to the truly *infinite* quantum that relates to itself in a higher power of itself.[23] Yet, since they are not *finite* quanta either, they lack the "being-for-self" that characterizes the simple quantum, namely being-for-self in the form of a determinate, internally manifold "one" (SL 167-9 / LS 211-13). They are, indeed, quantitative determinacies, but such that have ceased being self-relating "ones" and slipped back into the pure form of being-for-one or being-a-moment. dy and dx are thus no longer self-relating quanta at all, either finite or truly infinite. They are quantitative, but as purely relational and "qualitative" – quantitative determinacies that are altogether *non-quanta*, rather than *quanta* that are not only quanta but also infinitely self-relating (and "momentary").

Yet just as, in the sphere of quality, "being-for-one" relates to itself in its counterpart, which is also "being-for-one", and so constitutes a new being-for-self in the form of the one (*Eins*), so too dy and dx as mere moments constitute together one self-relating unit.[24] $\frac{dy}{dx}$ must thus be regarded "as a single indivisible sign" (SL 228 / LS 292), and in this sense it is truly infinite, self-relating quantity after all (even though it is not self-relating through raising itself to a power of itself).[25] Moreover, due to its self-relating character, $\frac{dy}{dx}$ is also a single finite quantum (i.e. 2x, $3x^2$ and so on) which has, or can be given, "a completely determinate quantitative value" (SL 229 / LS 294). *As* this finite quantum, however, $\frac{dy}{dx}$ is no longer a *qualitative* relation as such – that is, a qualitative quantitative relation – but rather the exponent of such a relation; or, as Hegel puts it, "this quantum of the whole does not concern the relation of its moments, *the nature of the matter*, i.e. the qualitative determination of magnitude" (SL 213 / LS 272). The mathematical infinite, $\frac{dy}{dx}$, is thus a finite

quantum that expresses the relation between qualitative moments of quantity that are themselves not quanta at all.

At the end of the last section I asked: between which items is $\frac{dy}{dx}$ a relation? We now know that it is a relation between items that are nothing but moments of that relation. More will need to be said to make fully clear what this means. What is already clear, however, is that, for Hegel, $\frac{dy}{dx}$ is not a relation between quanta that are infinitely small (or between quanta at all). It is a relation between "*quantitative determinacies*" (SL 206 / LS 263) – whose precise nature has yet to be clarified – that are completely relative to one another and in that sense "qualitative". Hegel arrives at this conception of the differential coefficient by reflecting on the logical structure of the true quantitative infinite, and he takes himself to be invited to do so by the familiar idea of the infinitesimal itself. Tradition has it (or at least did so, with some exceptions, up to and during Hegel's lifetime) that dy and dx are infinitely small magnitudes or "differences" (see SL 223 / LS 285), but in Hegel's view the idea of the infinitely small is contradictory. He is directed by the very idea of this "infinite", however, to the *true* quantitative infinite and he finds there the conceptual basis for an alternative conception of $\frac{dy}{dx}$ – a conception that is no longer vulnerable to the objections raised against the infinitely small. Hegel's conception of the true quantitative infinite is itself derived in the course of speculative logic; accordingly, even though his conception of the mathematical infinite is not so derived, he can claim that he has justified the latter through the philosophical "concept" (SL 204 / LS 259).

$\frac{dy}{dx}$, FRACTIONS AND SERIES

Hegel highlights the distinctive character of $\frac{dy}{dx}$ by comparing it with another relation between quantitative items, in which the latter are both moments *and* quanta at the same time. This relation is that of the fraction (*Bruch*). A fraction, Hegel notes, is not a simple whole number, but a quantum or number "determined through the mediation of *two other numbers* which stand to one another as amount and unit, the unit itself being a determinate amount" (SL 208 / LS 265-6). So in the fraction $\frac{2}{7}$, 2 is the amount and 7 the unit, and the quantum comprises *two* sevenths (rather than three or four).

In this fraction, Hegel maintains, the two sides of the relation are, first, "indifferent quanta" that have an independent identity outside of their relation: 2 and 7 form the fraction $\frac{2}{7}$, but they can also stand alone as whole numbers in their own right (SL 209 / LS 266). Second, however, they are also *moments* of the relation that constitutes the fraction. As moments, they count not simply

as 2 and 7, but as quantities *relative to one another* (or, as Hegel puts it, "according to their determinacy *with respect to one another*"). That is to say, they count as moments of the direct *ratio* they form. As such, they can be replaced by other quanta, such as 4 and 14 or 6 and 21, without altering the fraction itself. Insofar as 2 and 7 are moments of the fraction $\frac{2}{7}$, therefore, their character as "indifferent quanta" – their "indifferent limit" – is *negated* or "sublated": for they are present *not* just as 2 and 7, but as replaceable moments. With this negation, Hegel writes, "they begin to have a qualitative character"; indeed, "they have the moment of infinity in them" (SL 208-9 / LS 266). This is not to deny that, as mere moments, 2 and 7 lack the genuine infinity that consists in being self-relating: as mere moments, they exhibit the quality of being-for-one, not that of being-for-self. Nonetheless, they may be said to contain "the moment of infinity", insofar as they are no longer simple quanta, but precisely qualitative *moments* in relation to one another.

In the ratio of powers an infinite quantum is qualitative in both ways: in being self-relating (or "being-for-self") and in being a moment of that self-relation (or "being-for-one"). Such a quantum remains a finite quantum, x. Yet its infinity consists, not in its being a quantum as such, but in its relating to itself in its second power (and in being a moment of that self-relation). Furthermore, the relation-to-self present in the relation of x to x^2 does not belong uniquely to the quantum x (rather than y or z). It is the purely qualitative form of "infinite" self-relation that can be exhibited by any finite quantum.

The fraction $\frac{2}{7}$, which is a direct ratio, is also a finite quantum that is qualitatively infinite. It exhibits the "moment" of infinity, insofar as its constituent components are not just indifferent quanta but precisely relational *moments*. It also exhibits infinity (in the fuller sense) as the *relation* between those moments – the relation in which "being a moment" relates to "being a moment" and so constitutes infinite *being-for-self* (in accordance with the logic of being-for-one in the section on quality).[26] In these respects the infinity of the fraction – even when not squared – resembles that of the quantum raised to a power of itself. In another respect, however, the fraction differs significantly from the quantum raised to a power. In the latter, as just noted, the relation between a quantum and its power is purely qualitative and does not belong uniquely to that quantum or any other quanta: whatever quantum we take, it can always be "for itself" in the power to which it is raised.[27] In the fraction or direct ratio, by contrast, the relation does belong uniquely to the quanta in the relation: for, although 2 and 7 can be replaced by 4 and 14, the ratio between 2 and 7, which is the fraction itself, must be preserved. The specific quanta 2 and 7 thus determine that the ratio constitutive of the fraction cannot be, for example, 3 to 8 or 5 to 13. Accordingly, the fraction exhibits qualitative infinity in an "imperfect" manner, since such infinity – its constitutive ratio – is still

specific to the indifferent quanta that make up the fraction (and so has a fixed quantum as its exponent) (SL 209, 214 / LS 266, 274).

This "imperfection" is removed in the differential coefficient $\frac{dy}{dx}$ (even though the latter lacks the fully developed quantitative infinity found in the ratio of powers). This is because the moments of the ratio are no longer quanta at all but merely qualitative moments of quantity. The mistake made by those who regard dy and dx as infinitely small *quanta* is thus that they confuse $\frac{dy}{dx}$ with a *fraction*, when the two are in fact logically distinct.

In the course of discussing the structure of the fraction Hegel acknowledges that it can also be expressed as a decimal *series*: $\frac{2}{7}$, for example, can be expressed as 0.285714 . . . (SL 209 / LS 267). He also notes that in cases (such as $\frac{2}{7}$), in which such a series recurs endlessly, one might consider that recurring series to be the *infinite* expression of the fraction and the fraction itself to be the *finite* expression of that infinite series. He points out, however, that such "infinite" expressions of fractions are merely examples of the "bad infinity" of the infinite progress (SL 210 / LS 268).[28] True infinity, in the sphere of quality, consists in relating to oneself in the other and so not being limited by that other; and in the sphere of quantity, such infinity is realized most fully in the quantum that relates to itself in a power of itself. Compared to this true infinity, Hegel claims, the "infinite *series*" is actually a "*finite expression*" of a fraction, since it is simply an "incomplete aggregate", indeed an aggregate that is perpetually incomplete (SL 211 / LS 269-70). Hegel admits that such an "infinite" series is commonly taken to be "something lofty and exalted", but he dismisses such a judgement as merely "the superstition of the understanding" that lacks comprehension of the true infinite.

The decimal expressions of fractions that do not consist of endless series (such as 0.25 as the expression of ¼) are also finite, insofar as they express the fraction as a mere limited quantum or amount. By contrast, Hegel maintains, what is often regarded as the "*finite expression*" of the amount concerned, namely its expression as a *fraction*, is in fact "the truly *infinite expression*" (SL 211 / LS 269). This is because the amount is expressed as a *relation* between (albeit imperfectly) qualitative *moments*, which, as the relation of moment to moment, or negation to negation, is also a "self-relating unity". This is not to deny the importance of decimals in calculation; Hegel insists, however, that infinity in the philosophical sense is to be found not in such decimals but in their corresponding fractional, relational, expression.[29]

Yet it is not only fractions that can be understood in terms of a series. Hegel maintains that Isaac Newton employed the "*form of the series*" (*Reihenform*) to derive the differential coefficient, and that Lagrange later took up Newton's "method of the series" once again (SL 225, 227, 259 / LS 287, 291, 335).[30] Hegel recognizes that Lagrange set aside the idea that dy and dx are infinitesimals, and he commends him for conceiving of $\frac{dy}{dx}$ as simply the first derived function of an original function. He notes, however, that Lagrange still regarded dy and

dx as "*increment[s]*", and thus as quanta, and he remains critical of Lagrange's method of the series (SL 223, 227 / LS 285, 291).[31]

Lagrange defines a function as a "quantity formed in whatever manner from another quantity".[32] He then seeks the differential coefficient or "derivative" of a function, f (x + i), by expanding the latter into a *series* of derived functions.[33] In so doing, in Hegel's view, Lagrange conceives of the expanded function simply as the sum of the members of that series. This is apparent in his famous theorem which states (in Hegel's wording) that "the difference" – Lagrange's "i", or the more familiar dx – "without becoming zero, *can be taken to be so small that each term of the series exceeds in magnitude the sum of the following terms*" (SL 227 / LS 291).[34] For Lagrange, therefore, on Hegel's reading, each derived function is essentially a quantum that is part of a series (or sum) – a quantum whose successors can be ignored on the basis of their "relativity" (*das Relative*), that is, if their relative smallness permits it (even though they are not to be conceived as "infinitely" small). For Hegel, by contrast, $\frac{dy}{dx}$ is not essentially a quantum, or part of a series, but it is a distinct qualitative *relation* in its own right (though, *pace* Hartmann, it can have a sum as its value).[35] This relation, Hegel concedes, is found by expanding the binomial, $(x + dx)^n$, into a series of terms; but it is completely expressed by the first term that arises through that expansion, without regard to the other terms, and is thus not essentially the first part *of an ongoing series* (SL 226 / LS 290).[36] Like the fraction, therefore, $\frac{dy}{dx}$ is conceived to be genuinely "infinite", when it is understood, not as part of a series or as a series itself, but as a unique relation – a relation (in the case of $\frac{dy}{dx}$) between items that are neither infinitesimals, nor finite increments, but purely qualitative moments of quantity (see SL 215 / LS 275).

$\frac{dy}{dx}$ AS THE RELATION BETWEEN THE "ELEMENTS" OF QUANTITIES

In Hegel's view, dy and dx are quantitative determinations that have the form of purely relational, qualitative moments. As I indicated above, however, they are, for Hegel, also qualitative in a further sense (2: 217). More specifically, they have quality, rather than some quantum or amount, as their *content*: they are – or stand for – certain qualities *of* quanta, rather than quanta as such. We must now consider this point in more detail.

In lines cited above Hegel writes that "dx, dy are no longer quanta, nor are they supposed to signify quanta, but they have meaning only in their relation, *a meaning merely as moments*" (SL 215 / LS 275). In the mathematical infinite, $\frac{dy}{dx}$, therefore, the quantum is "not just sublated as this or that quantum, but as quantum as such" (SL 215 / LS 276).[37] Hegel then goes on to note that "what *remains* the principle is *quantitative determinacy* as the *element* of quanta, or, as has also been said, the quanta in *their first concept*". So $\frac{dy}{dx}$ is not a relation

between quanta themselves, but it has as its "principle" their "element" and so (we are to assume) is the relation between one such element and another. What then is the "element" of a quantum?

We find a clue in another passage in the first remark on calculus. There Hegel writes that Newton, too, conceives of dy and dx – which Newton calls "moments" – as each "the quantum [. . .] as it is in its *becoming*, in its *beginning* and *principle*, that is, as it is in its *concept*", and Hegel goes on to equate the latter with the quantum "in its qualitative determination" (SL 219 / LS 280).[38] This suggests that the "element" of a quantum, which in the first passage is also identified with the quantum in its "concept", is precisely the quantum in its "qualitative determination". In other words, the "element" of a quantum is the *quality* of that quantum, its distinctive character rather than its amount. In yet another passage, Hegel confirms this suggestion by equating a quantum's element explicitly with its quality. While discussing (and countering) the view that it is always illegitimate to identify an arc with its tangent, Hegel concedes that "the *arc* is indeed *incommensurable* with the *straight line*", and he gives as the reason that "its element is first of all of another *quality* than the element of the straight line" (SL 232 / LS 297).

$\frac{dy}{dx}$, then, is a relation between two qualitative moments, each of which is (or signifies) the element or quality of a different quantum. More specifically, Hegel claims, $\frac{dy}{dx}$ is the relation between the "element of the ordinate" and the "*element of the abscissa*" (SL 231 / LS 296). So what does this mean?

If we take a function, say $y = x^3$, and plot the values of this function in a Cartesian coordinate system, these values form a curve. The *ordinate* is the y-value of a point on the curve that corresponds to the same value on the vertical y-axis; and the *abscissa* is the x-value of the point that corresponds to the same value on the horizontal x-axis. The ordinate and abscissa as simple *quanta* obviously stand in the constant relation to one another that is expressed by the function: in our example the y-value is always the cube of the x-value. The *increments* of the two values are also related to one another, since an increase in one value produces an increase in the other, though this relation is not constant but changes as the values get bigger.[39] According to Hegel, however, the "elements" or qualities of the two variables, y and x – of the ordinate and abscissa – also stand in a constant relation to one another, and it is this relation that is expressed by $\frac{dy}{dx}$. In $\frac{dy}{dx}$, therefore,

> the element of the ordinate [. . .] is not to be taken as the *difference of one ordinate from another ordinate*, but is rather the difference or *qualitative* determination of magnitude in relation to [*gegen*] the *element of the abscissa*. [. . .] The difference, insofar as it is no longer a difference of finite magnitudes, [. . .] has collapsed into simple intensity, into the determinacy of one qualitative moment of a ratio in relation to the other.
>
> —SL 231 / LS 296

Yet this still doesn't explain fully what the "element" of a quantum is. Hegel's point becomes clear, however, in words omitted from the last quotation. In his view, the "difference" or relation between the elements of the ordinate and the abscissa is to be understood as "*the principle of the one variable magnitude with respect to [gegen] that of the other*". Accordingly, the element or quality of a quantum is the principle of its variability: *how* it changes in relation to the way the other changes.

Now in one respect the distinctive "how" of two magnitudes, x and y, in a function, $y = x^n$, is already visible in the changes they undergo in accordance with that function: they change *in such a way* that y always equals x^n. In this respect, however, the magnitudes relate to one another as *quanta*, so their distinctive "how" actually consists in a series of different quanta and their increments; it does not appear in the explicit form of a "*how*" or quality. This occurs only insofar as the principles or qualities of the magnitudes stand in immediate relation to one another *without the mediation of quanta* – the relation that Hegel identifies with $\frac{dy}{dx}$. Only in this relation do the qualities of the magnitudes find expression *as qualities*.

The differential coefficient, $\frac{dy}{dx}$, as Hegel conceives it, thus now has a clear meaning. It is the constant relation, derivable from a function $y = x^n$, between the qualities of the two variable magnitudes, or the *ways* in which they change with respect to one another, in abstraction from the specific amounts of the variables and their increments.[40] This relation is itself expressed by a quantum that has the form nx^{n-1}, and so it has a "determinate quantitative value" (SL 229 / LS 293-4). Yet the relata are neither quanta, nor increments, but principles of variability or ways of changing, conceived explicitly *as qualities*.

It is important to note, however, that these qualities are not knowable as qualities independently of one another. They are knowable only *in* their relation to one another, indeed only *through* this relation. The qualities of two magnitudes in a given function become visible *as* qualities – in abstraction from the quantitative increments produced by the original function – only through the fact that they stand in *this* constant relation, $\frac{dy}{dx}$, to one another. If $y = x^3$, then x and y change in such a way that they take on a specific series of quanta: 2 and 8, 3 and 27, and so on. Yet only in the relation, $\frac{dy}{dx}$, whose exponent is $3x^2$, do the relative qualities of the two variables become visible as such – namely *as* the qualities, or ways of changing, that yield *this* constant ratio, rather than another.

To repeat: the qualities or "elements" of the two variables, x and y, are not determinable as qualities – as *ways* of changing or "principles of variability" – outside of the ratio $\frac{dy}{dx}$. They are visible *as* qualities only *in* this ratio. The qualities become apparent as such in the fact that they produce *this* constant ratio with *this* exponent, rather than another: $3x^2$, say, rather than $2x$ or $4x^3$.

This ratio will, of course, itself produce a series of different quanta, as x takes on different values; but it is not in that series of quanta that the qualities of x and y become visible as qualities. They become visible as such in the ratio itself – the ratio between moments *that are no longer quanta*.

Hegel states that nothing more can be said, or needs to be said, about $\frac{dy}{dx}$. It is simply the first qualitative relation that can be derived from a given function, and "there is otherwise no real meaning [*kein reeller Sinn*] to ascribe to $\frac{dy}{dx}$" (SL 250 / LS 322). As we shall see later, it acquires "real meaning" only when it is understood to be equal to another, more specific ratio. At that point it will become clear what it means in *concrete* terms to say that the qualities of x and y in the function, $y = x^n$, stand in the constant ratio to one another, $\frac{dy}{dx}$: for we will see that x and y change in relation to one another *in such a way* that the slope of the tangent to the curve formed by that function is always $\frac{dy}{dx}$ (2: 240-1).

NEWTON AND EULER

Hegel considers the concept of the differential coefficient, set out in his remarks on calculus, to be a new one, based on the true quantitative infinite. Yet he also contends that this concept has "hovered before" (*vorgeschwebt*) the mind of others who have tried to justify and determine the character of the mathematical infinite (SL 206 / LS 262).[41] Indeed, he claims, the "thought of the matter" (*Gedanke der Sache*) implicitly underlay their conceptions of the latter, though they were not able to render it explicit "in its determinacy" or to grasp it "as concept" (SL 217, 222 / LS 277, 284).[42] This is true especially of Newton and Euler.

Hegel is famously critical of aspects of Newton's mechanics and his theory of colour; but with regard to the mathematical infinite, $\frac{dy}{dx}$, he maintains that "the thought [*Gedanke*] cannot be more correctly determined than as *Newton* has stated it" (SL 217 / LS 277).[43] Strictly speaking, of course, this is misleading, since (from Hegel's point of view) the thought *can* be more correctly determined: namely, as he himself has done in his remarks. Nonetheless, Hegel thinks that Newton comes very close to that correct determination with his concept of "fluxions".

In his *Principia* (1687), Newton considers certain quantities to increase or decrease "as if by a continual motion or flux". He calls the "instantaneous increments or decrements", by which such quantities grow or diminish, "moments", and he gives the name "fluxions" to the speeds at which these quantities change, to the "velocities of increments and decrements".[44] A fluxion, for Newton, is thus a ratio between a "moment" and time – e.g. so much per second – and so is equivalent to "the $\frac{dx}{dt}$ or $\frac{dy}{dt}$ of the Leibnizian calculus".[45] Hegel, however, sets aside Newton's ideas of "motion and velocity" and sees

in fluxions Newton's equivalent to the mathematical "infinite", or $\frac{dy}{dx}$, as such (SL 217 / LS 277).

There is a slight ambiguity in Hegel's interpretation of Newtonian fluxions, since he appears to regard them both as moments (or "*vanishing divisibles*") *and* as ratios between moments (or as the "*limits*" of such ratios) (SL 217 / LS 278). Furthermore, he draws for his interpretation on two passages from the *Principia*, one of which does, but one of which does not, mention "fluxions".[46] Hegel's aim, however, is not to provide a close textual analysis of Newton's use of the term "fluxion". It is, rather, to show that Newton's ideas implicitly express, and so anticipate, the correct "thought" of the mathematical infinite.

In *Principia* 1, i, Lemma 11, Scholium (in which there is no mention of fluxions or moments), Newton formulates the idea of the "ultimate sums and ratios of vanishing quantities". Such quantities, he insists, are "not indivisibles but evanescent divisibles"; and since they are "evanescent", their ultimate sums and ratios cannot be of or between "definite parts", but must be those in which the quantities disappear.[47] Hegel makes direct reference to this idea and also highlights Newton's defence of it against possible criticism. As both Hegel and Newton note, one could object that (in Hegel's words) "vanishing magnitudes do not have a *final ratio*, because the ratio before they vanish is not the last, and when they have vanished there is no longer a ratio" (SL 217 / LS 278). Newton's response, however, is to point out that the ratio of vanishing magnitudes is to be understood, not as the ratio before they vanish or after they vanish, but as "the ratio *with which* they vanish" – the ratio they have *as* they vanish. Similarly, "the first ratio of nascent quantities is that *with which* they begin to exist [or come into being]".[48]

According to Hegel, Newton's conception of an ultimate ratio implicitly expresses the proper thought of the mathematical infinite because such a ratio holds between magnitudes "in their vanishing", and thus (in Hegel's interpretation) between magnitudes that are "no longer quanta" but equally *not just nothing* (SL 217 / LS 278). They are "no longer quanta" precisely because they are *ceasing* to be divisible magnitudes; yet they are not just nothing because they are still present *as* ceasing to be. The ratio between such magnitudes is thus not a ratio between definite "parts" or quanta: as Newton puts it, "those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities".[49] An "ultimate ratio" is rather the limit ratio in which quanta are *and yet are not*. Indeed, it is "the limit of the quantitative ratio" in which that ratio itself both is and is not (SL 217-18 / LS 278).[50] (Similarly, the moments of "first" proportions are not yet "finite particles" or definite quanta, but nor are they nothing. This is because they are "the just-now nascent beginnings [*principia jamjam nascentia*] of finite magnitudes".)[51] Note that Hegel does not directly endorse this conception of an "ultimate" (or of a "first") ratio. Yet he takes it to correspond *implicitly* to the mathematical infinite as he conceives it: to the quantitative ratio in which the terms are not quanta, but are not just nothing at

all, because they are "*qualitative* moments of quantity" (SL 218 / LS 278, emphasis added). For Hegel, Newton's ultimate ratios are logically – though not explicitly – ratios between quantitative determinations, each of which is "purely a moment of the ratio" and for that reason qualitative (SL 218 / LS 279).

Hegel notes that Newton does not himself take the logical step from "vanishing magnitudes" to "qualitative moments of quantity" (and grasp the "concept" of the mathematical infinite) because he continues to think of the "moments" in a final or first ratio as "decrements" or "increments" (SL 219 / LS 280-1). Such moments, for Newton, are the decrements *with which* quantities disappear or the increments *with which* they begin, and so are not themselves definite finite quantities.[52] In Hegel's view, however, the very idea of an "increment" (or "decrement") – and thus of a Newtonian "moment" – "fall[s] *within* the category of the immediate quantum" (SL 219 / LS 281). Newton's conception of a "moment" of the final or first ratio thus remains the contradictory one of a *quantum* that is *not* a quantum. He comes close to the correct "thought" of the mathematical infinite; but he is prevented by the ideas of "*increment*, *growth*, increase" from "extracting the determination of the qualitative moment of quantity in its purity [*rein*] from the representation of the ordinary quantum".[53]

Leonhard Euler is another mathematician who, in Hegel's view, implicitly expresses the correct thought of the mathematical infinite but fails to grasp it explicitly. Euler's failing, however, is not so much that he holds on to the idea of an increment, but that he does not draw the necessary logical conclusion from his assertion that an infinitely small quantity is, or will be, nothing, "nil" (SL 221 / LS 282).

For Euler, an infinitely small quantity is a "disappearing" (*evanescens*) quantity and so will in truth be "= 0". Accordingly, "dx = 0", and the ratio between two "evanescent" magnitudes, dy and dx, is "0 : 0".[54] How, though, can there be any *ratio* between two magnitudes that are nothing? And how can we distinguish them at all as "dy" and "dx"? Euler answers this question by distinguishing between an "arithmetic" and a "geometric" ratio. Arithmetically, Euler acknowledges, there is no difference between 0 and 0 and so the one cannot stand in a "ratio" to the other. Yet they can be conceived as standing in a ratio to one another, if that ratio is understood to be a geometric one that is proportional to another such ratio. So, in Hegel's words,

> if 2 : 1 = 0 : 0, then, because of the nature of the proportion, since the first term is twice as great as the second, the third must also be twice as great as the fourth; according to the proportion, 0 : 0 is thus to be taken as the ratio of 2 : 1.
>
> —SL 222 / LS 283[55]

Hegel regards Euler's explanation as unsatisfactory, since "a nil no longer has any determinacy at all" and so cannot stand in any ratio to another (SL 221

/ LS 283). What interests Hegel more, however, is the fact that Euler does not draw the obvious logical consequence – obvious to Hegel, at least – from his own insight. Euler maintains that an evanescent quantity is (or will be) nothing – "nil" – and so he concedes that it is not a quantum, or, as Newton puts it, that it is not a "finite particle".[56] Yet, at the same time, he recognizes that such a nil, such a non-quantum, is not nothing whatsoever but is the *moment of a ratio*. He conceives of that ratio, however, as proportional to a ratio between *quanta*: so $0 : 0 = 2 : 1$. Accordingly, he fails to see that a quantity that is no longer a quantum but that is still a relational moment must, logically, be a *qualitative* moment of quantity. As Hegel puts it, Euler "goes as far as the negative of the quantum, giving to it definite expression" – by claiming that both dx and dy "= 0" – but he "fails to grasp this negative at the same time in its positive meaning as qualitative determinations of quantity" (SL 221 / LS 283).

Euler, like Newton, thus remains a thinker of the "understanding", who does not conceive of quantity explicitly as qualitative. Neither recognizes explicitly that "the infinitely small, which in the differential calculus occurs as dx and dy, does not have the merely negative, empty meaning of a non-finite, non-given magnitude [. . .], but has the specific meaning of the qualitative determinacy of the quantitative, of a relational moment as such" (SL 228 / LS 292). Like Newton, however, Euler *implicitly* conceives of quantity as qualitative – and so implicitly comprehends the "thought" of the mathematical infinite – by formulating the idea of quantity that is not a quantum or "finite particle" but that is nonetheless a moment of a ratio.[57]

$\frac{dy}{dx}$ AS A DERIVED FUNCTION OR "DERIVATIVE" (OF A POWER-FUNCTION)

Hegel's account of $\frac{dy}{dx}$ rests on the concept of the true quantitative infinite. Otherwise, however, that account can be regarded as deflationary: for Hegel conceives of $\frac{dy}{dx}$, *without* the familiar thoughts of the "infinitesimal" or the "increment", as the relation between two qualitative *moments* of quantity, each of which is itself the quality of a variable magnitude, x or y, or its way of changing with respect to the other. Hegel understands this relation to be derived from, and so inherent in, a determinate, finite function – such as $y = x^3$ – that connects the two variable magnitudes. The purpose of differentiation, for Hegel, is thus "the reduction of the finite function to the qualitative relation of its quantitative determinations" (SL 221 / LS 283).

As we have seen, this qualitative relation is constant and is expressed by a quantum derived from the function concerned. So if $y = x^3$, the ratio $\frac{dy}{dx}$ is $3x^2$, however x and y *as quanta* change with respect to one another. This quantum, $3x^2$, is itself a function and so is the first "derived function" (*abgeleitete*

Funktion) of the "original" finite function. The peculiar purpose of differentiation is thus not only to "reduce" a function to the "qualitative relation of its quantitative determinations", but thereby also to find the particular quantum that expresses that relation and that is the first "derivative" of the function (see SL 223, 229, 242 / LS 285, 294, 312).[58]

In this latter respect, Hegel's conception of differential calculus is close to the modern one. It differs from the modern conception, however, by restricting the most appropriate use of calculus to a specific kind of original function. Hegel notes that mathematics introduced the "infinite", $\frac{dy}{dx}$, into the analysis of "*functions of variable magnitudes*" – magnitudes indicated by x and y – as opposed to functions with fixed magnitudes, such as $3 + 2 = 5$. Yet he goes on to point out that the term "variable magnitude" is actually too vague to identify the specific functions that "higher analysis" is interested in (SL 213-14 / LS 273). Such analysis is (or should be) concerned primarily with functions whose component variables are, or include, magnitudes *raised to a power*, or what Hegel calls "*power-determinations*" (*Potenzenbestimmungen*) (SL 215, 236-7 / LS 275, 304). More precisely, Hegel states, "at least one" of the magnitudes in a function or equation should be "in a *higher power* than the first" (SL 238 / LS 305). This qualification is necessary because, in Hegel's understanding, the first power – x^1 – is, strictly speaking not a "power" at all: as he puts it, "the first power is a power only in relation to higher powers", but "by itself, x is simply some indeterminate quantum" (SL 239 / LS 307).

Why then should calculus deal above all with functions involving quanta raised to a power? The philosophical reason is as follows. In a linear function, $y = ax$, two independent *quanta*, y and x, are in relation to one another. The values of y and x can, of course, vary, but each nonetheless stands for a simple, indifferent quantum. Furthermore, the relation between the two is itself fixed by an independent *quantum*, a: for if $y = ax$, then $\frac{y}{x} = a$ (SL 214 / LS 274). In the function, $y = x^2$, however, things are different. First, the number, x, raised to a power, x^2, is no longer just a simple quantum, but a quantum that relates to itself in that power: for the latter is the quantum multiplied by *itself*. The quantum raised to a power, therefore, exhibits "the moment of infinity, of being-for-self, that is, of being determined through itself", and so is a "qualitative determinacy of magnitude" (SL 236 / LS 304). Second, the number, y, is not just a simple quantum either, since it is directly determined *by* the qualitative quantum x^2. y considered on its own is, indeed, a simple (unspecified) quantum; in the equation, however, it is no longer a simple quantum, but is a side or moment determined by the other moment, which is qualitative. Accordingly, y is itself "qualitative" (though as a moment, rather than as being-for-self). Third, the relation between y and x^2 is in turn no longer fixed by a simple quantum, but is the "*qualitative* relation" between a power and a moment determined by it (SL 214 [ll. 33-5] / LS 274 [ll. 21-3]).

The relation between y and x^2, however, implicitly contains a further derivative relation between the "elements" or qualities of y and x. That is to say, as y changes its quantum in proportion to x^2, and thereby produces a curve in the Cartesian coordinate system, the *qualities* of the ordinate, y, and abscissa, x – that is, *how* each changes with respect to the other – also remain in a constant ratio to one another that is distinct from the ratio expressed by the original function, $y = x^2$ (see 2: 223-4). This constant ratio is in turn expressed by a definite quantum, and the task of discovering the latter falls to differentiation.

In Hegel's view, therefore, it is clear why calculus deals principally with functions containing powers: for its distinctive task is to find the derivative *qualitative* relation – expressed by a quantum – that is inherent in an original *qualitative* function. In Hegel's own words, "since differential calculus specifically operates with qualitative forms of magnitude, its peculiar [*eigentümlich*] mathematical subject matter must be the treatment of power-forms [*Potenzenformen*]" (SL 236 / LS 304).

Yet surely one can differentiate a linear function just as easily as a qualitative one involving powers: if $y = ax$, then $\frac{dy}{dx}$ is a. Given Hegel's conception of $\frac{dy}{dx}$, such differentiation shows that a qualitative relation between the elements of the ordinate, y, and abscissa, x, can also be derived from an original linear function. In the case of a linear function, however, that qualitative relation coincides with the original relation between the quanta, y and x; that is to say, the constant ratio between the *ways* in which each changes with respect to the other coincides with the ratio between the two changing *quanta* themselves. This coincidence is evident, mathematically, in the fact that, if $y = ax$, then $\frac{dy}{dx}$ is a *and* $\frac{y}{x} = a$ (SL 239 / LS 307); and philosophically, the coincidence reflects the fact that in a linear function the components have no qualitative dimension to them beyond the "quality" of being simple *quanta* as such.

In Hegel's view, therefore, one can indeed differentiate linear functions, but there is no particular point in doing so since there is no distinct derivative to be discovered: $\frac{dy}{dx}$ proves to be the same ratio as $\frac{y}{x}$. This is not to deny the importance of differentiating the linear components of "qualitative" functions, such as $y = x^3 + 2x$; but "there is no point [*keinen Sinn*] in differentiating *for their own sake* the equations $y = ax + b$ (the equation of the straight line), or $s = ct$ (that of simple uniform velocity)" (SL 239 / LS 307).[59] For Hegel, a reason for thinking that it *is* appropriate to differentiate linear functions is provided by the "implicitly correct demand" that the method of calculus be "*generalized*" (SL 215, 252-3 / LS 275, 326). Hegel insists, however, that this demand should not lead us to overlook the principal, and distinctive, purpose of calculus. As he writes, "we would, indeed, have been spared much formalism in the consideration and treatment of these matters, if it had been perceived that the calculus is concerned not with variable magnitudes as such but with *power-determinations*" (SL 215 / LS 275).

As Michael Wolff points out, Hegel does not deny that the range of differentiable functions extends beyond that of power-determinations and "in actuality is immeasurably large", including not just linear functions but also "the hyperbolic function $y = \frac{a}{x}$".[60] Yet Hegel is not seeking a *general* concept of the derivative, or $\frac{dy}{dx}$, for all cases. He is seeking to discover the derivative's "distinctive" (*eigentümlich*) character, its specific conceptual structure, and his claim is that, whatever its various mathematical uses may be, it is *logically* (from the perspective of philosophy) a qualitative relation between quantitative moments. This conceptual structure is most explicit in functions derived from originals that are themselves qualitative (SL 238-9 / LS 307).[61] As we have seen, a quantum becomes qualitative or "for itself", in Hegel's view, by being raised to the second *power* or "squared" (and higher powers, for him, are mere continuations or repetitions of the square).[62] It is, therefore, in discovering the derivatives of *power-functions* that calculus performs its distinctive, philosophically justified role, whatever other roles it may also play in mathematics.[63]

FINDING DERIVATIVES BY EXPANDING (OR "DEVELOPING") BINOMIALS

How, then, are such derivatives to be discovered? We can, of course, just employ the technique of differentiation and move straight from the function, $y = x^n$, to the function, $\frac{dy}{dx} = nx^{n-1}$; but that does not *prove* that the latter is the first, and necessary, derived function of the former. So how do we prove that?

We do so, according to Hegel, through the expansion or "development [*Entwicklung*] of a binomial", namely, the binomial that the original function – and specifically the "power-determination" it includes – is taken to be (SL 230, 242 / LS 295, 312). How, though, is it possible, given Hegel's point of view, to conceive of this power-determination as a binomial? On the familiar interpretation of calculus this is not a problem. We start with the function $y = x^n$, and we then take both y and x to be increased by a certain amount, namely dy and dx respectively. The function is thus recast as $(y + dy) = (x + dx)^n$, and so both sides of the equation are conceived as binomials. Hegel, however, rejects the idea that dy and dx should be conceived as "increments" or any kind of quanta, and he thinks of them as qualitative moments of a ratio. He cannot, therefore, turn x^n into a binomial by understanding x to be *increased by dx*.

Hegel has other reasons, however, for conceiving of x^n as a binomial. First, a power is a number and thus a *sum*, and so can be "analysed into an arbitrary amount of numbers", including into two. So 4^2 is 16, which can be broken down into 2 + 3 + 5 + 6, or into 7 + 9. Second, and more importantly, "if the

power" – x^n – "is taken as a sum, then its radical number, the root" – x – "is also taken as a sum" (SL 240 / LS 308), and as such this root, too, can be regarded as a binomial – a + b – without the need for any additional "increment". The root can, of course, also be thought as a polynomial, a + b + c + d, but Hegel maintains that increasing the number of its components beyond two is "a mere *repetition* of the same determination" (SL 240 / LS 309). The important idea, in his view, is that a root, as a number, can be thought as a sum of *two* numbers.

Once the root of a power is conceived as a binomial, the power can then be expanded, yielding a "system of members"; so, in Hegel's notation, "$x^n = (y + z)^n = (y^n + ny^{n-1}z + \dots)$" (SL 241 / LS 310).[64] These members in turn each have a specific character or "*qualitative determinacy*" that directly "results from the *raising to a power*" – or "*potentiation*" – "of the root taken as a sum". Accordingly, they are "wholly *functions of the potentiation and of the power*" (SL 240 / LS 309). These members are thus proven to be derived functions of the original power by being produced (or discovered) by expanding that original power itself, once the latter – or its root – has been conceived as a binomial.

Yet Hegel is aware that there is a danger in treating a power, or its root, as a *sum*, since it leads us to conceive of the expanded power as a sum, or series of sums, too, and so risks making us think of the derived functions of the original function as principally parts of that sum or members of that series (SL 240 / LS 309). In Hegel's view, however, the derived function of an original should not be conceived essentially as a sum or as part of a sum (though it can have a sum as its value), but rather as a *ratio* (2: 213). Furthermore, he maintains, what gives the derivative its significance is not its place in an ongoing series, but the relation it has to its original.[65] All the functions produced by the expansion of a power (conceived as a binomial) are functions of that original power; yet Hegel claims that each function can also be understood as the first derivative of the function that precedes it, which is thus the "original" for that derivative. The series of functions is thus not to be conceived as a *series*, but rather as the *repetition* of the single relation between derivative and original: "the relation between all of them" – the members of the series – "is the same; the second function is derived from the first in exactly the same manner as this is derived from the original, and for the function counted as second, the one derived first is again the original" (SL 242 / LS 312).[66]

The derivation of one function from another is a little more complicated than Hegel acknowledges here (as we will see in the next section). Nonetheless, what he says makes it clear that he does not want to understand an expanded binomial power as principally a series or sum; and this in turn moves him to modify subtly the idea that the original power, or its root, should itself be

conceived as a sum. He thus suggests that "from the sum one should take up only the *relation* [*Beziehung*]" between the numbers into which the root of a power can be decomposed (SL 241 / LS 310).[67] As he puts it, "the relation as such of the magnitudes is, on the one hand, what remains after abstraction is made from the *plus* of a sum as such, and on the other hand, what is required for finding the functions of the expansion of that power".

In Hegel's view, therefore, what is required to find the derived functions of a power is simply to conceive of that power, or its root, as having the *form* of a relation between two quantitative items, the *form* of a binomial, and then to expand that power; we do not need to think of the power as actually being a sum. Of course, Hegel abstracts the form of a binomial from the thought of a power or root as a sum, which thought thus comes first. Yet he suggests that we could even dispense with that thought: for the idea that the root of a power is a "relation" is already contained in the very form of the equation in which the power appears. An equation such as $y^n = ax^m$, we are told, renders each side relational because it shows each to be a function *of the other*. In so doing, it shows each to be more than just itself and in that sense to have "a plus in it". Accordingly, "their character of being functions of one another gives them this determination of a *plus*" (SL 241 / LS 310).

When we understand the root of a power merely to have the *form* of a binomial, the "plus" that is taken to belong to the root is itself conceived as a mere form, *not* as an "increment" or quantum. By contrast, in the "usual analytical development" of a power "an increment, dx, i" is taken to be added to the variable magnitude that is its root and the "power of the binomial" is then expanded. In Hegel's view, the "usual" idea that the derivatives of a function are to be discovered through such expansion is correct. "The so-called increment", however, "is not supposed to be a quantum, but only a *form*, whose whole value is that it *aids* the development" (SL 241 / LS 311). Accordingly, Hegel argues, one should think of that form as just a "factor" or "external means for the development", and one should represent it simply with "1 (the one") (SL 242, 259 / LS 311-12, 335). This "factor one", he maintains, enables the root of a power to have the form of a binomial, but, in contrast to dx or i, it does not introduce the idea of an increment or quantum (which can then be thought to be infinitely small).

The insertion of the "factor one" into the root of a power to turn it into a binomial may look arbitrary, but it is not. It is justified by the fact that that root is itself a *relation* – for the reasons noted above – and it simply renders that relation explicit. The expansion of the power, conceived as a binomial, thus discloses the derivative relations that are inherent in the original relational power. In other words, that expansion shows the power to be, or contain, "*a system of relational determinations*" (SL 239 / LS 308). It is these derivative

relations, rather than infinitesimal magnitudes, that, for Hegel, are the true subject of differential calculus.

DERIVED FUNCTIONS AS "COEFFICIENTS"

As we have seen, Hegel equates the "members" (*Glieder*) of an expanded or developed binomial power with the "*functions of the potentiation and the power*", that is, with the derived functions of the power (SL 240 / LS 309). This equation is, however, inexact, since derived functions, as Hegel conceives them, are in fact found only in the "coefficients" of the members of an expansion (as he notes in several places).[68] These coefficients are the parts of each member that contain the original variable, x, that is the root of the power concerned. So, if we take the function, $y = x^4$, regard the root as a binomial such that $(y + dy) = (x + dx)^4$ – retaining the familiar notation rather than Hegel's "1" – and then expand the right side of this equation, we get:

$$x^4 + 4x^3.dx + 6x^2.(dx)^2 + 4x.(dx)^3 + (dx)^4$$

We can eliminate x^4, since it is cancelled by y on the left side of the equation, so the first member of the expansion is $4x^3.dx$. The *coefficient* of this first member, however, is just $4x^3$, and it is this, rather than the whole member, that is the first derivative, $\frac{dy}{dx}$, of the original function, $y = x^4$.

Note that at one point Hegel suggests that we not refer to the coefficient of "*the first term* of the development", since that term is "*first* in relation to the other terms following it in the series" (SL 242 / LS 312). Similarly, he later describes the object of calculus as the "so-called first function" (SL 253 / LS 327). The point of these remarks, however, is not to deny that the expansion of a power yields a series of derived functions. It is to highlight the fact – as Hegel sees it – that a derived function should be understood, not principally as occupying a place in a *series* (as first, second or third), but in relation to its original function, that is, as the derivative *of that original*. Furthermore, as we have seen, Hegel argues that each derivative serves as an original for the next derivative (2: 232). Each derivative beyond the first is thus derived not just from the original function, but also from its own original (which is itself a derivative). The series of derived functions is thus not merely a *series*, but rather the repetition of the relation between the original and its first derived function.

When one looks closely at the expansion of a specific function, however, this understanding of the derived functions beyond the first is hard to sustain: for, strictly speaking, the subsequent derivatives, understood as coefficients, do not have the *same* relation to their predecessors as the first derivative – the first coefficient – has to the original function. *Pace* Hegel, no coefficient after the first is the direct derivative or "differential coefficient", $\frac{dy}{dx}$, of its predecessor.

Take the function we have just been considering: $y = x^4$. When $(x + dx)^4$ is expanded, we get

$$x^4 + 4x^3.dx + 6x^2.(dx)^2 + 4x.(dx)^3 + (dx)^4$$

If we then extract the “coefficients”, we have

$$4x^3 + 6x^2 + 4x$$

The first coefficient in this set is, indeed, the derivative of the original function; but it is obvious that the second is not the derivative of the first and the third is not the derivative of the second, since the derivative of the first is $12x^2$ (not $6x^2$) and the derivative of this is $24x$ (not $4x$). So is Hegel simply mistaken in his understanding of the functions of potentiation? No; but we have missed out something important that is made clear in the work of Lagrange, which Hegel knew well. This is that one has to match the binomial expansion of a function with what is known as the Taylor series in order to discover from that expansion the true derivatives of that function (beyond the first).

The Taylor series, in Lagrange’s notation, links the derivatives of a function as follows:

$$f(x+i) = fx + f'x.i + \frac{f''x}{2}.i^2 + \frac{f'''x}{2.3}.i^3 + \frac{f^{iv}x}{2.3.4}.i^4 + \ldots$$

Here *f* means “function of”; *f* then means “(first) derivative of *f*” (that is, $\frac{dy}{dx}$); *f* means “derivative of *f* ”, or second derivative of *f* (that is, $\frac{d^2y}{dx^2}$), and so on; and i is equivalent to dx.[69] The derivatives are thus contained in the following coefficients of the members of the series:

$$f'x + \frac{f''x}{2} + \frac{f'''x}{2.3} + \frac{f^{iv}x}{2.3.4}\ldots$$

If we now match this series of coefficients with the series of coefficients from the expansion of the binomial function $(x + dx)^4$, namely

$$4x^3 + 6x^2 + 4x$$

we can find all the derivatives of this latter function. So

$$f'x = 4x^3 = \frac{dy}{dx}$$

$$\frac{f''x}{2} = 6x^2;\ \text{therefore } f''x = 12x^2 = \frac{d^2y}{dx^2}$$

$$\text{and } \frac{f'''x}{2.3} = 4x; \text{ therefore } f'''x = 24x = \frac{d^3y}{dx^3}$$

The coefficients of the binomial expansion of a function, such as $y = x^4$, thus do, indeed, give us the derivatives of that function beyond the first, just as Hegel contends – but only if those coefficients are interpreted in terms of the Taylor series.

By the way, once the derivatives of a function have been found in this way, Hegel's claim that each is derived from its predecessor by repeating the *same* relation or operation is seen to be justified. First, each has the same form nx^{n-1} in relation to its predecessor; but second, each is derived in the same way by regarding its predecessor as a binomial power and expanding it. So if we take $4x^3$ and regard it as a binomial, we get $4\,(x + dx)^3$. If we then expand this function, we get

$$4\,(x^3 + 3x^2.dx + 3x.(dx)^2 + (dx)^3).$$

If we then remove the original function, $4x^3$, and extract the coefficients, we find that the first derived function is $4 \,.\, 3x^2$ or $12x^2$ (which is correct).

If we then take $12x^2$ and regard it as a binomial, we get $12\,(x + dx)^2$. If we now expand this function, we get

$$12\,(x^2 + 2x.dx + (dx)^2)$$

If we again remove the original function, $12x^2$, and extract the coefficients, we find that the first derived function is $12 \,.\, 2x$ or $24x$ (which is also correct).

No doubt Hegel should have made it clearer that expanding a binomial function does not by itself directly yield the derivatives of that function beyond $\frac{dy}{dx}$ (that is, $\frac{d^2y}{dx^2}$ and so on). Yet, if one bears in mind Hegel's interest in and partial indebtedness to Lagrange, one can see the justification for his claim that such expansion does yield these derivatives.[70]

$\frac{dy}{dx}$ AS A RELATION WITH NO "REAL MEANING"

To recapitulate: Hegel begins his consideration of calculus by examining what he takes to be the usual procedure for deriving $\frac{dy}{dx}$ from a function, $(y + dy) = (x + dx)^n$, and pointing to two problematic features of that procedure. The first is that dy and dx are conceived as infinitesimals, even though the very idea of an "infinitesimal" is contradictory. The second is that, when the function is expanded, all terms containing powers of dx (beyond the first) are "ignored" for being vanishingly small, and this leaves us with the equation, $\frac{dy}{dx} = nx^{n-1}$.

This exact result is thus reached through a procedure that creates "the appearance of *inexactitude*" by ignoring terms of the expansion deemed "*negligible*" (SL 205 / LS 261).

Hegel's new conception of dy and dx eliminates the first problem by regarding them, not as infinitesimals or indeed as quanta at all, but as "qualitative moments of quantity" (SL 218 / LS 278). This in turn has a bearing on the second problem: for members of the expansion that include higher powers of dx can no longer be neglected on the basis of being vanishingly small. Yet Hegel does not deny that deriving $\frac{dy}{dx}$ from a binomial function does, indeed, require us to "disregard" (*außer acht lassen*) all terms of its expansion except the first (after x^n) (SL 228 / LS 291). The reason we have to do so, however, is that those other terms *do not belong to* $\frac{dy}{dx}$ *in the first place*.

For Hegel, the first derivative or "differential" of a function is a qualitative relation between quantitative moments that is "wholly exhausted by the first term of the series which results from the expansion of $(x + dx)^n$", or more precisely by the "coefficient" of the first term (SL 226, 230 / LS 290, 294). Subsequent terms thus form no part of that derivative, and no "appearance of inexactitude" is created by neglecting them. Similarly, the coefficient of each subsequent term completely expresses the derivative of its predecessor (provided that the expansion is understood in relation to the Taylor series). Each coefficient, properly understood, thus expresses a new derivative – that is, a new relation – that is qualitatively distinct from the others: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, and so on. Accordingly, the terms of a binomial expansion "have a determinate *qualitative significance*, and terms are disregarded, not because they are insignificant in magnitude, but because they are qualitatively insignificant" to a particular derivative (SL 228 / LS 291).

It should also be emphasized that, although the derivatives are discovered through the generation of a series, they are not to be thought of principally as moments *of* a series (or as parts of a larger sum); they are simply qualitatively *different* functions produced by repeating the relation between the original function and its immediate derivative (in the manner described above). In this respect, Hegel's conception of derived functions differs from that of Lagrange, as Hegel understands it, even though the former is indebted to the latter.

We have not, however, reached the end of Hegel's account of calculus, for we still need to look more closely at Hegel's claim that $\frac{dy}{dx}$, beyond being the first qualitative relation that can be derived from a given function, has "no real meaning" (*kein reeller Sinn*) (SL 250 / LS 322). This claim can itself be understood to have two related meanings.

First, Hegel is claiming that, despite appearances, differentiating an equation does not yield another real equation but merely a relation. If we differentiate the equation of a curve, there arises, or appears to arise, a new equation; so, to use Hegel's example, if we differentiate $x^2 - ax - b = 0$, we get $2x - a = 0$.

Hegel points out, however, that "it is by no means self-evident that such a derivative equation is also correct", since the terms of the original equation have been replaced by their derivatives and their "value" has thus been altered (SL 249 / LS 322; see also SL 245 / LS 316). Indeed, he claims, the result of differentiation is not "truly" (*wahrhaft*) an *equation* at all, but merely a *relation* or *ratio* (*Verhältnis*) derived from the original equation (SL 244 / LS 314).[71] Differentiating $x^2 - ax - b = 0$ thus in fact just yields the function $2x - a$, which is the exponent of a ratio, and differentiating $px = y^2$ just yields the ratio $p : 2y$ (SL 245-6 / LS 317). Similarly, "all that the equation $\frac{dy}{dx} = P$ expresses is that P is a *ratio*, and otherwise no real meaning can be ascribed to $\frac{dy}{dx}$" (SL 250 / LS 322).[72] The derivative of a function lacks "real meaning", therefore, because it does not present a new equivalence or equation, but is simply a relation or ratio inherent in the original function (or, *qua* quantum, it is simply the exponent of such a ratio).

Now, as we know, this derivative ratio is a relation, expressed by a quantum, between two *qualitative* moments of quantity. More precisely, it holds between the "elements" of the ordinate and the abscissa in a coordinate system, that is, between their *ways* of changing with respect to one another, conceived explicitly as "ways" or qualities, not as series of varying quanta (SL 231 / LS 296). The derivative is thus not a ratio between the *magnitudes* of x and y themselves; that ratio is expressed in the original function. Nor is the derivative (as is commonly held) a ratio between increments to x and y (whether these are considered to be infinitesimal or not). Neither of these ratios, therefore, can give "real meaning" to the derivative. The latter is simply the ratio, preserved by the ways in which x and y change with respect to one another, in which those "ways" become visible as *qualities* without the mediation of a series of quanta.

Hegel adds that the derivative, $\frac{dy}{dx}$, is a "linear" (*lineares*) ratio (SL 246 / LS 317). A linear equation is one in which the terms, whether constants or variables, are quanta without powers, e.g. $y = 2x$, and which thus produces a straight line, rather than a curve, when its values are plotted on a coordinate system. A linear ratio, by contrast, is – or rather corresponds to – a ratio *between* straight lines that can be expressed by quanta with or without powers. In calling $\frac{dy}{dx}$ a "linear" ratio, therefore, Hegel points out that it is a ratio "with which certain lines are in proportion" – straight lines that belong to the "system determined by the curve" produced by the original function (SL 246 / LS 317-18). He immediately adds, however, that "*with this, nothing is yet known*": for we do not know *which* lines of the curve's system are "in proportion" to the ratio expressed by the derivative. The "interest" of analysis thus centres, not just on finding the derivative of a function, but also on discovering which particular lines connected with the function's curve have the same ratio to one another as the *derived ratio*; that is to say, it lies in finding the "equality of two ratios" (SL 246 / LS 317).

So, to repeat: the derivative of a function containing powers is the ratio between the "elements" or *qualities* of the ordinate and abscissa. Yet it is also a "linear ratio" that matches that between certain lines. These lines are not specified by the linear ratio itself and so remain undetermined; we thus do not know directly from that derivative ratio which lines are in proportion to it and so "to what other ratio it is equal" (SL 250 / LS 322). It is, however, only "such an equation, *proportionality*" with another ratio that "gives a value and meaning" to a derivative ratio. This is the second sense, therefore, in which the derivative of a function, considered by itself, has "no *real* meaning".

Hegel maintains that several kinds of object can be understood by means of derivatives. Geometrical squares and cubes, whose equations involve powers and from which "linear determinations" can be derived, can be understood in this way, as can accelerated motion (see SL 243 / LS 313). The simplest example, however, is provided by "curves determined by an equation of the second power" and the straight lines connected with such curves, namely, the "tangent, subtangent, normal, and so on" (SL 244 / LS 314).[73] More specifically, Hegel notes, the ratio between certain of these lines can be shown to match the ratio that is the derivative of the function that generates their parent curve. When this happens, the latter ratio itself acquires "real meaning". In particular, it becomes clear that the ratio preserved by the *ways* in which the ordinate and abscissa, y and x, change with respect to one another is in fact the ratio between the ordinate and the subtangent *as lines* in the system of the curve concerned.[74]

In Hegel's view, therefore, there are two distinct parts to the method of analysis. First, the "theoretical or general part" consists in "the finding of the *first function*" – the first derivative ratio – "from the given equation of a curve". Second, the "applied" part consists in "finding those lines belonging to the curve [*an der Kurve*] that stand in this ratio" (SL 246 / LS 318). According to Hegel, however, it is only Lagrange who distinguished clearly between these two stages and so set analysis "on the truly scientific path". Furthermore, it was only Lagrange who, once he had determined how to find derivatives, actually sought to *discover* which lines stand in the corresponding ratios. Earlier mathematicians, Hegel claims – including Newton's teacher, Isaac Barrow – just assumed from the outset that the ratio sought through differentiation (or its precursor) matches that between specific lines connected to a curve. They thus not only understood that ratio to be between an "*increment of the ordinate*" and an "*increment of the abscissa*" (rather than between qualitative moments of quantity), but their method rested on the simple "*assumption of the proportionality* of the increments of the ordinate and abscissa to the ordinate and the subtangent" (SL 245 / LS 316).[75]

Hegel objects to the question-begging approach of mathematicians, such as Barrow, but he recognizes that they correctly identified the lines whose ratio

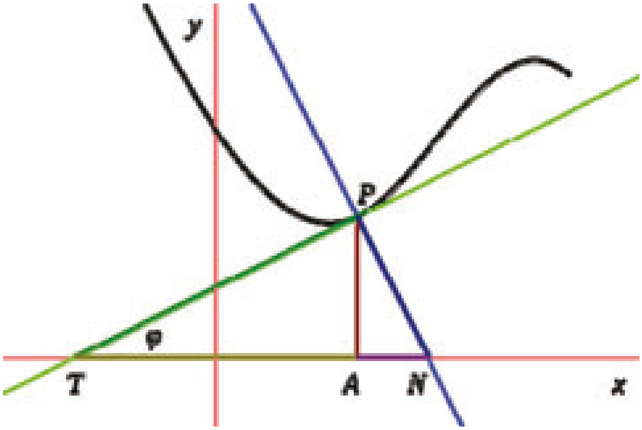

FIGURE 1

matches the derivative ratio of a function: namely, the *ordinate* and the *subtangent*. The tangent to a curve is a line that touches the curve at a point, and the ordinate is the line drawn from that point to the x-axis and at right angles to that axis (in contrast to the normal which is also drawn from that point to the x-axis but which is at right angles to the tangent). The subtangent is then the line drawn from the base of the ordinate on the x-axis to the point on that axis at which it is cut by the tangent. In the diagram above, the tangent (or a portion of it) is PT, the ordinate PA, the subtangent TA and the normal PN.[76]

If the tangent to a curve cuts the x-axis at the base of the y-axis, where both x and y = 0, the subtangent is the same length as the abscissa considered as a line drawn horizontally from the y-axis to the point P at which the tangent touches the curve. Otherwise, the subtangent is longer or shorter than the abscissa-line, depending on whether the tangent cuts the x-axis to the left or to the right of the y-axis. The ratio between the ordinate and the subtangent is thus conceptually distinct from that between the ordinate and abscissa considered as lines, and it is only the *former* ratio that is "in proportion" to the first derivative ratio of a function that produces a curve.

Now the ordinate and subtangent of a curve form the sides of a right-angled triangle that are, respectively, *opposite* and *adjacent* to the tangential angle, φ, and the hypotenuse of this triangle is formed by the tangent. The ratio between the ordinate and the subtangent, or more precisely the length of the ordinate

divided by that of the subtangent, is thus tan φ, which is the *slope* of the tangent at the point it touches the curve. This in turn means that the derivative, $\frac{dy}{dx}$, of the curve's function – to which the ratio between ordinate and subtangent is "in proportion" – also determines the slope of the tangent at that point.[77] This, therefore, can be regarded as the "real meaning" of the ratio, $\frac{dy}{dx}$. In Hegel's view, however, $\frac{dy}{dx}$ does not have this meaning purely by virtue of being the derivative of a function that generates a curve. It *acquires* this meaning only when it is matched with *another* ratio, namely that between ordinate and subtangent (SL 246, 250 / LS 317, 322).

The qualitative relation, $\frac{dy}{dx}$, that is derived from a function, $y = x^n$, thus acquires a "real meaning" by being equated with the slope of the tangent to the curve produced by that function (SL 246-8 / LS 318-19). This means in turn that the specific *qualities* of x and of y in the function, or the characteristic *ways* in which each changes with respect to the other, become visible *as* qualities – in abstraction from their various magnitudes, increments and decrements – in the constant ratio between the ordinate and subtangent of the function's curve. Note that this word "constant" is important. The particular magnitude of the ratio, and thus of the slope of the tangent, will of course change as x and y change their magnitudes; it will, however, always be equal to nx^{n-1} and it is in this unchanging determination that the qualities of x and y in the function manifest themselves as such. So if, for example, $y = x^2$, then x and y change *in such a way* that the ratio between the ordinate and subtangent, $\frac{o}{s}$, remains 2x, whatever specific increments are added to x and y. In that constant ratio, therefore, we see *how* x and y change with respect to one another *without* the mediation of the different magnitudes they take on. By contrast, as noted above (2: 224), what we see in the original function is the way x and y change precisely *by* taking on a specific series of different magnitudes: 2 and 4, 3 and 9, and so on.[78]

Hegel thus succeeds in making the differential coefficient, $\frac{dy}{dx}$, intelligible without any reference to "infinitesimal" magnitudes. In this derived function itself such magnitudes are replaced by qualitative determinations of quantity; and the other ratio, with which $\frac{dy}{dx}$ is "in proportion", is one between lines with definite finite magnitudes (and subject to definite increments or decrements). Infinitesimal magnitudes thus play no role at all, and a possible reason for criticizing differential calculus is thereby removed.[79]

One might object that Hegel takes a long time – either 60 or 80 pages, depending on the German edition – to reach a familiar conclusion: namely, that $\frac{dy}{dx} = \tan \varphi$. (For Hegel, by the way, this is a genuine equation, since it equates two ratios.)[80] His aim, however, is not to lead to new mathematical insights, but to eliminate what he takes to be certain *conceptual* misunderstandings in the familiar conception of calculus. In particular, he wants to replace the idea that $\frac{dy}{dx}$ is a relation between infinitesimal magnitudes with the idea that it is a relation between *qualitative* moments of quantity. Then, however, he needs

to explain how a ratio between qualitative moments can be used to understand the relation between finite *magnitudes* (such as the lines connected with a curve). To do this, he follows Lagrange in distinguishing between the theoretical part of analysis, concerned with finding the derivative of a function, and the applied part, in which that function is used to understand specific objects (though he criticizes Lagrange for failing to conceive of derivatives as qualitative relations) (SL 227-8 / LS 291). Hegel does not seek, therefore, to take calculus forward mathematically. In my view, however, his conceptual clarification of the foundations of calculus is a significant achievement.[81]

PART TWO

Measure

CHAPTER ELEVEN[1]

Specific Quantity

In the first *Critique* Kant counts among the conditions of the objects of experience the categories of "quantity" and of "quality", but he does not derive one set of categories from the other *logically* (see CPR B 106, 202-18). In the *Logic*, by contrast, Hegel argues that quantity is made necessary logically *by* quality, specifically by the qualitative category of the "one" (*Eins*) and the connected categories of "repulsion" and "attraction" (which, in quantity, become "discreteness" and "continuity") (SL 132-45 / LS 166-84).[2] Hegel goes on to argue that quantity in turn makes quality necessary and thereby gives rise to the explicit unity of the two in the form of *measure* (*Maß*). Measure, he notes, was a central concept for the Greeks, who, indeed, maintained that "*everything has a measure*" (SL 284 / LS 367). It is, however, a concept that is lost on Kant, as it is lost on Spinoza (SL 282-4 / LS 364-7).[3] Hegel's speculative logic thus restores the concept of measure, neglected by the moderns, to its rightful place in our understanding; and it does so by demonstrating that measure is made necessary logically by the very quantity and quality that for Kant – in the guise of two "classes" of categories – are the indispensable conditions of objective cognition.

Now in his account of measure in the *Logic* Hegel discusses numerous natural phenomena and laws, many of which were unknown to the Greeks, and he associates each one with a particular form of measure. These phenomena and laws include specific heat, specific gravity and Kepler's third law of planetary motion.[4] Hegel's interest in these aspects of nature is not accidental, but follows from the fact that "the different forms in which measure is realized also belong to *different spheres of natural reality*" (SL 286 / LS 369).

It is important to emphasize, however, that Hegel's conception of measure is not itself guided by his understanding of nature or by the findings of modern science. His derivation of the various forms of measure proceeds *immanently*

by rendering explicit what is implicit in the concept of measure itself. ("The whole course of philosophising", Hegel states in the *Encyclopaedia*, is, indeed, "nothing else but the mere *positing* [*Setzen*] of what is already contained in a concept" [EL 141 / 188 [§ 88 R].) Natural phenomena and laws are then adduced as examples of the measures that have been derived logically. Speculative logic is understood by Hegel to be both a logic and an ontology or metaphysics: an account of the necessary categories of thought and of the fundamental ways of being.[5] The examples from nature thus serve to confirm that the measures made necessary by logic belong just as much to the world.

Logic, then, does not follow nature or natural science, but nature exemplifies the structures derived a priori by logic.[6] If we are to understand why there are measures in the world, therefore, it is crucial that we understand the distinctive *logic* of measure that makes its various forms necessary. My aim in this chapter and the ones that follow is to explain, as clearly as I can, how that logic proceeds.

MEASURE AS A SPECIFIC QUANTUM

As we have seen, a quantum that raises itself to a higher power relates to *itself* in becoming another quantum, and in relating to itself in this way it exhibits the *quality* of being-for-self (SL 278 / LS 359). In being quantitative, therefore, it constitutes qualitative being. A quantum, however, is truly for itself, and so qualitative, only when it is a wholly self-relating quantum – self-relating without becoming another – that is, when it constitutes quality *by itself*. As such, the quantum is what Hegel calls a *measure* (*Maß*). A measure is thus "the simple relation [*Beziehung*] of the quantum to itself" (SL 288 / LS 371).

Since a quantum-as-measure constitutes quality by itself, and so by relating immediately to itself, it constitutes quality in the form of simple immediacy (rather than the more developed form of being-for-self we encountered in 1.1.3). Moreover, since it stands alone – at least, initially – the measure is itself a simple, immediate quantum. A quantum, however, has its own determinacy and so is a number (see SL 168-9 / LS 212-14). A measure is thus not only a *self-relating* quantum, but also one that is *determinate* in its self-relation. The quality it constitutes by virtue of being self-relating – being *itself* – must, therefore, also be determinate. To begin with, a measure is thus simply "some determinate [*bestimmtes*] quantum" that constitutes "some determinate quality" (SL 288 / LS 371). This quality, however, does not exist on its own, but is the quality of *something* (in accordance with the logic of *Dasein* [see SL 88-9 / LS 109-10]). A measure, therefore, is some specific amount, or "specific quantum", that confers a specific quality on something and without which the thing would lose that quality. The term "specific quantum" will in turn denote, from now on, not just a particular quantum, but one that is constitutive of (or, at least, specific to) quality.

Note, however, that the quality of something is not just one property among many, but that which gives a thing its determinate character: it constitutes its determinate being. This quality is internally complex in having an affirmative and a negative dimension and in encompassing the thing's determination and constitution, but it is nonetheless *the* quality of the thing that makes it what it is: a wood or a meadow or a football pitch. As such, the quality can undergo change, but in so doing it also preserves its identity and the identity of the thing with which it coincides – until, that is, it reaches its limit, at which point any further change causes the thing to cease to be and become something else (see SL 92, 101 / LS 114, 125-6). A measure, therefore, does not just give something one of many properties, but confers on it the defining quality "through which something is what it is" (SL 280 / LS 361).

Conceived as a measure, the quantum is thus no longer a limit to which a thing is indifferent (SL 288 / LS 371). A mere quantum can be changed without destroying the thing concerned: the latter can become bigger or smaller and remain what it is.[7] The quantum (or range of quanta) that constitutes a thing's *measure*, however, cannot be changed without destroying the thing, because it is precisely what gives the thing, or enables it to have, its distinctive quality: it "belongs", and is specific, to that quality (SL 291 / LS 374).[8] In this sense a measure is an unchanging, fixed quantum (or range of quanta), which perhaps explains why Hegel calls it "magnitude determinate *in itself*" (*die* an sich *bestimmte Größe*) (SL 291 / LS 375).

The idea that something must preserve a particular size or degree to be what it is is familiar from everyday experience: water, for example, must be kept below 100° C or it turns into steam, and "a republican constitution like the Athenian, or an aristocratic constitution mixed with democracy, is possible only in a state of a certain size" (SL 287 / LS 370).[9] What interests Hegel above all, however, is the logical structure of a measure as such. The latter is a quantum, and so is external to itself in the ways we have seen in previous chapters. Yet it constitutes quality because it is explicitly "self-relating externality". Yet it is also determinate in its self-relation. Logically, therefore, the measure is the quantitative "determinacy that has returned into simple equality with itself" and so is "one with determinate *Dasein*" – or quality – "just as the latter is one with its quantum" (SL 288 / LS 371).

Speculative logic proves that being cannot just be indeterminate but must take the form of determinate, finite things. Such things must also have a certain quantity or "magnitude". We have now learned that a thing must also have a certain measure: a specific quantum (or set of quanta) that allows it to be what it is. Indeed, it is only when a thing's measure is understood that we understand fully what it is to *be* that thing: measure, as Hegel puts it, is "the *concrete truth of being*" (SL 285 / LS 367). Accordingly, Hegel writes, if we want to derive a proposition from the point we have now reached in the logic, we can say that

"*all that exists* [*da ist*], *has a measure*" (SL 288 / LS 371). "Every determinate being [*alles Dasein*]", he continues,

> has a magnitude, and this magnitude belongs to the very nature of a something; it constitutes its determinate nature and its being-within-self [*Insichsein*]. Something is not indifferent to this magnitude, as if, were the latter to alter, it would remain what it is; rather, the alteration of the magnitude would alter its quality. As a measure the quantum has ceased to be a limit which is none; it is from now on the determination of the thing, so that the latter would perish [*zugrunde ginge*], if it were increased or diminished beyond this quantum.
>
> —SL 288-9 / LS 371

Note, by the way, the subtle ambiguity contained in these lines. A measure is a magnitude that "constitutes" (*ausmacht*) the determinate nature or quality of something; so that nature or quality is determined *by* that magnitude. Yet the magnitude equally "belongs to" (*gehört zur*) the nature of the thing, and so is just as much determined *by* the latter, *by* the thing's quality. This ambiguity arises from the fact that a measure is in truth the immediate *unity* of quantity and quality. A measure is, indeed, the magnitude that enables something to be this thing rather than that; but only because it belongs to the *quality* of this thing to be sustained by that magnitude and so to have this measure. This ambiguity should be borne in mind throughout the logic of measure, as it helps explain why sometimes quantity and sometimes quality is the principal determining factor in a given form of measure – though the decisive reason why this is the case will always reside in the specific form itself.

Hegel does not draw attention to this ambiguity, but he does point out that measure is ambiguous in a further sense – a sense that will drive the logic of measure forward. Note that a measure is not an ideal standard that a thing has to meet in order to be itself; it is the magnitude a thing *actually has* and to which it owes its distinctive quality. This magnitude must be – or fall within the range of – the thing's measure, for if the thing *has* it, it obviously permits the thing to be what it is. The (further) ambiguity of such a magnitude is this. As a simple *quantum*, it can change like any other: it is "an indifferent magnitude" that is capable of increase and decrease (SL 289 / LS 372). As a *measure*, however, it grounds, or makes possible, the quality of the thing. This in turn sets a *limit* to the extent to which it can change: for if it is to preserve the quality concerned, it can change only within a certain range. The measure is thus a thing's magnitude or quantum that is "distinct [*verschieden*] from itself as quantum" and limits the latter (SL 289 / LS 372). The actual temperature of the water before us is a quantum that can vary from 1° C to 99° C. This temperature range enables the water to remain a liquid and so constitutes its

measure. Yet that range is also just a range of *quanta*, and as such can be increased or decreased beyond the limits of the measure that it is. If this happens, however, water will change its quality and become steam or ice. The quantum (or range of quanta) that something actually has thus constitutes both the bare, changeable quantum of the thing *and* the measure that sets a limit to the changes that quantum can undergo.[10]

Note that the measure *as measure* (rather than mere quantum) is itself quantitative and so changeable: it is a variable limit. This does not, however, prevent it from being the limit it is; it simply means that the measure is not, or is not necessarily, restricted to one quantum but can encompass a range of quanta (as I have anticipated in previous remarks). The quantum limited *by* the measure can thus increase or decrease and still remain within the limit imposed on it, just as water can change its temperature and remain liquid. Insofar as it remains within that limit, therefore, the quantum of a thing is a matter of indifference to it.[11] If, however, the quantum increases or decreases beyond the limit set by the measure, the quality attached to that quantum will cease to be, and the thing will become an altogether different thing (with a different measure).

The measure of a thing, however, cannot itself prevent the thing's quantum from exceeding its limit: for that quantum as such is irreducibly changeable, and the measure cannot take this feature away from it. The measure does not, therefore, make it impossible for the quantum to go beyond the limit set for it, but simply determines that, if it does so, the quality disappears. In this sense, a thing's magnitude, which allows the thing to be what it is and so is or belongs to its measure, is actually impotent *as a measure* in the face of its own quantitative nature: as a measure, it sets a limit for itself as mere quantum, but in so doing it cannot prevent itself from going (as a mere quantum) beyond that limit. That limit is thus both *quantitative* and *qualitative* at the same time: it can always be exceeded by the thing's magnitude, but only at the cost of destroying the thing's quality.

It follows from the nature of measure, therefore, that a thing can change its quantity without altering its quality, but that it will (or may) reach a point at which that quality, and therewith the thing, ceases to be. This change in quality and demise of the thing, Hegel writes, will appear "*unexpected*", if one is unaware of the thing's measure, because it will seem that only a change in quantity is occurring. One can then be tempted to try to make such change in quality intelligible with the help of the idea of "*gradualness*" (*Allmählichkeit*): one can imagine that simple changes in the magnitude of a thing *gradually* produce a change in its quality, and thus a new thing, before our very eyes (SL 289-90 / LS 373). Yet Hegel insists that thinking of qualitative change as "gradual" in this way reduces it to something purely quantitative – to a mere matter of increase or decrease – and so actually makes it impossible to explain how any *qualitative* change could occur. In Hegel's view, the only thing that can

explain how a change in quantity brings about a change of quality is the *measure* of a thing; understanding the latter, however, requires us to give up the desire to explain everything in purely quantitative terms by means of the idea of "gradualness".[12]

In this context Hegel briefly discusses the ancient Greek "sorites" paradoxes (from the Greek for heap, *soros*), attributed to Eubulides of Miletus.[13] "The question was put", Hegel writes, "does the plucking of one hair from the head or from a horse's tail produce baldness, or does a heap cease to be a heap if one grain is removed?" (SL 290 / LS 373). The answer, surely, is no, and it continues to be no when one pulls out a second and then a third hair, or removes a second and then a third grain. Indeed, formal logic tells us that the answer should remain no, as long as one continues to remove just one item at a time. Yet, of course, we eventually reach a point at which we say that the head is bald or there is no more heap; so at that point pulling out a single hair makes us bald after all, leaving us with a paradox.[14]

Hegel insists that such paradoxes are neither "an empty or pedantic joke", nor merely sophistical as if the contradiction they contain were a "pretense", but that they are "in themselves correct" (SL 290-1 / LS 374). They are, in other words, not just puzzles to be solved or dissolved with the resources of formal logic, but paradoxes that disclose a fundamental truth. This truth is the truth of *measure*, namely, that quantitative changes are not merely quantitative, but at some point – or within a certain range of points – make a *qualitative* difference. The value of the paradoxes, therefore, is that they expose the "mistake" of "assuming a quantity to be only an indifferent limit". As Hegel notes, those who think that repeatedly removing just one grain should not eliminate the heap (before there are no grains at all) forget that "the individually insignificant quantities [. . .] *add up*" and that the sum constitutes a "qualitative whole" (SL 290 / LS 374). Similarly, those to whom the steady increase in their wealth "appears at first to be their good fortune" overlook the fact that such an increase may at some point lead to their misfortune, that is, to a reversal in the quality of their lives (SL 291 / LS 374).[15]

THE SPECIFYING MEASURE AND SPECIFIC HEAT

A measure is a quantum that constitutes (or enables there to be) a certain quality. A quality, however, does not stand alone, but is the quality of *something* (*Etwas*).[16] A measure is thus not just an abstraction, but a quantum constitutive of a *thing* with a certain quality, and it is as such that it differs from a "mere" quantum in the way we have seen.[17] Following the logic of "something", however, that mere quantum should itself be *something other* than the measure. Indeed, Hegel notes, this follows directly from the initial "immediacy" of the measure: the measure is immediately *itself* and must, therefore, be immediately

different from a quantum that is in turn immediately *it*-self. When this thought is rendered explicit, a new form of measure emerges. This new measure does not simply differ from the mere quantum that *it* is, but both sides now have "a distinct existence" (*eine verschiedene Existenz*) (SL 291 / LS 375).[18]

Yet, as we know, a measure is not indifferent to the mere quantum, but sets a limit to it and in that sense "negates" it. This continues to be true of the new measure: it, too, limits the mere quantum that lies outside it. It does so on the basis of its own specific determinacy and so proves to be the activity of *specifying* that external quantum. Measure has thus now to be understood, not just by itself, but in relation to an "alterable external" quantum, which it specifies (SL 291 / LS 375).

It should be stressed that what drives the logic of measure forward here is the double character of the first measure. On the one hand, this measure is the unity of quantity and quality: it is a quantum that constitutes and sustains a quality. On the other hand, quantity and quality remain *different* in the measure, since the latter contains quantity in two forms: once as constituting quality and thus as the measure, and once as a mere quantum. Moreover, the quantum as qualitative – as the measure – "negates" the mere quantum that it also is by setting a limit to it: this limit is one that that quantum cannot exceed without undermining the quality attached to the measure. Note that this difference between a quantum as measure and as mere quantum falls *within* the first measure itself: that measure sets a limit to the changes that *it*, as a mere quantum, can undergo (if such changes are not to destroy the thing). As a self-relating something, however, the measure now sets itself in relation to a quantum that is (or belongs to) something of its own and so falls outside the measure. In this way, the difference that is internal to measure mutates logically into a relation between a measure and *another* quantum. Such a relation is thus not an accidental feature of measure, but renders explicit the difference that is at the heart of measure from the start.[19]

A measure "in relation" first specifies the quantum it confronts by providing an external measure for it: one that Hegel calls a "rule" (*Regel*) or "standard" (*Maßstab*) (SL 291 / LS 375).[20] Since this rule and the external quantum are initially just immediately *other* than one another, the former does not actively negate and change the latter (as the third form of measure will do) but simply stands next to it. Yet, as a measure, the rule must specify and limit the quantum in some way. So how does it do so? We learn in the account of quantity that a quantum – or, more precisely, a quantum as a number – is a determinate "*amount*" of featureless "*units*" (SL 169 / LS 213-14).[21] The rule, therefore, must specify either the amount of the external quantum, or the units it comprises, or both. The amount, however, belongs to the quantum, since it makes the latter the quantum or number it is, and so it falls outside the rule. Accordingly, the rule – *unlike* the first form of measure – does not determine

how big something may be or what degree it may reach. It must, therefore, specify the quantum by providing the *unit* (*Einheit*) in terms of which the latter is to be measured. Now the rule, as a measure, is something specific and determinate, so the unit it provides for the quantum must also be determinate.[22] This unit is thus not just a bare unit as such, but a determinate one, such as a foot or a metre, and the quantum, which stands in relation to the rule, must in turn be a certain amount *of* such units. The rule specifies the quantum, therefore, by determining the latter to be an amount, not just of bare units, but of units of a *specific* character.

Note that, *pace* Burbidge, Hegel is here not just describing a process of measuring in which *we* engage.[23] He is arguing that being itself must produce measures and that these measures *themselves* serve to specify the magnitudes of other things. Yet insofar as they are no more than a rule or standard of measurement, such measures remain external to the quanta they specify. They can thus be replaced by other measures and so in that sense are "arbitrary" magnitudes (SL 289 / LS 372). (In Burbidge's words, "the quality being measured couldn't care less which standard has been adopted".)[24] Furthermore, since these measures are themselves *quanta*, they contain their own amounts. They can thus be specified in turn in terms of other units, just as a foot can be determined as an amount of inches.[25]

Hegel now points out, however, that measure must take a further, third form. This emerges as we continue to render explicit what is implicit in being a measure. As we have seen, a measure not only is the immediate unity of quantity and quality, but as such constitutes a *something* (*Etwas*). It must, therefore, be accompanied by, and directly related to, another something, and so, as Hegel puts it, it must have in it "this side of being-for-other" (SL 292 / LS 376). In accordance with the logic of "something", however, the first something must now also be open to being *changed* by the other to which it relates and so have what Hegel earlier, in the account of quality, called a "constitution" (*Beschaffenheit*).[26] Since the other is here principally another *quantum*, the something must be open in particular to having its quantum changed by that other.

Yet the first something is not merely a something, but also a *measure*. As such, it must limit and *specify* the quantitative change that the other brings about in it. In his account of quality Hegel argued that something is not completely at the mercy of other things, but has an intrinsic being or "determination" (*Bestimmung*) of its own that affects how other things affect and change it: "the determining from outside is at the same time determined by the something's own immanent determination" (SL 97 / LS 121). We now see more clearly one of the things that this means: through the measure that it contains, something limits in a specific way – and so specifies – the changes in quantity to which it is subjected by another.

The measure has thus mutated, logically, from an external standard or rule into an explicitly "*specifying measure*" (SL 292 / LS 376).[27] Accordingly, it now no longer relates only to a quantum that is outside and other than it, but *in* relating to another quantum it *also* relates to itself, to the quantum that it is: for it specifies the quantum within itself that comes from the other. In this respect the measure blends together the relation to another quantum, which characterizes the second form of measure, with the limiting of itself as a mere quantum, which characterizes the first form. This third form of measure thus embodies more explicitly than either of its predecessors what it is to be a measure.

Note, too, that its relation to the "mere" quantum is more *active* and *negative* than in the case of the rule. The rule simply limits such a quantum to being an amount of *these* units, rather than those; it thereby remains external to that quantum and leaves the latter itself unaltered. By contrast, the new specifying measure limits the change that is imposed on it by the other quantum, and thereby negates and changes that change: it alters the quantum that it is given by the other. In this way, the something constituted by the measure negates the mere quantum in two senses: it negates its own quantum insofar as the latter is determined by the other quantum, and so it negates that other quantum as well. It does so in a specific way that is governed by its own measure. In specifying the effect that another quantum has on it, therefore, the something shows itself to be something of its *own*, something *for itself*. Hegel pulls these thoughts together in the following lines:

> Inasmuch as something is a measure within itself, an alteration of the magnitude of its quality comes to it from outside, and it does not take on the arithmetical amount [*Menge*] of the alteration. Its measure reacts against it, relates as something intensive [*ein Intensives*] to the amount and assimilates it in a distinctive way [*auf eine eigentümliche Weise*]; it alters the externally imposed alteration, makes something else out of this quantum and through this specifying shows itself to be being-for-self [*Fürsichsein*] in this externality.
>
> —SL 292 / LS 376

It is crucial to recognize the complexity of the logical structure that Hegel sets out here. There is one something in relation to another, the first of which is a measure, whereas the second is merely a quantum (or, rather, something *insofar* as it is merely a quantum). The one that is a measure, however, is itself a quantum. As a measure, therefore, it stands in a negative relation to both its own quantum *and* that of the other: it negates its own quantum, insofar as the latter is in turn determined by the other. More precisely, it negates the amount that is *added* to its own quantum *by* the other. Here we see the clear difference between a merely quantitative relation between quanta and the more nuanced

relation between a quantum and a measure. If one bare quantum is added to another bare quantum, the latter increases by precisely what is added to it: add 2 to 3, and 3 becomes 5. Something with (or as) a measure, however, does not directly take on what is added to it: as Hegel puts it, it does not accept the "arithmetical amount" that is given to it. Rather, it accepts, and so increases by, an amount that has been specified by its measure. This additional amount remains a quantum, and is still dependent on the other quantum. Yet it is not completely dependent on the latter and is not a mere "quantum as such", but it is a quantum "specified in a constant [*konstante*] manner" (SL 292 / LS 376). The specifying measure, Hegel notes, thereby constitutes the "*exponent*" that governs the relation between something and any quantum that changes it. If the same amount is added to different things, therefore, their exponents ensure that each in fact increases by a *different* amount that is specific to it.[28]

To repeat: the specifying measure or exponent is the quantum, specific to a thing, that explains why different things increase by different amounts when the same amount is added to them. This is a new measure that is subtly different from, though related to, the first "specific quantum" that enables the thing to be what it is at all. That first measure, we recall, is a simple, immediate quantum (or range of quanta) that confers a specific quality on a thing. The new measure, however, is the explicit *negation* of the mere quantum, and so cannot be a mere quantum itself; it can thus no longer be a simple quantum like the first measure. What distinguishes a measure in general from a mere quantum is, of course, the moment of quality. The new measure or exponent must distinguish itself from a mere quantum, therefore, by being a quantum that is itself explicitly qualitative. A quantum is qualitative *as a quantum*, however, when it raises itself to a certain power (2: 219-20). What specifies the amount that is added to a thing is thus not just a simple quantum, but one that is a "*power-determination*" (*Potenzbestimmung*) – a square or cube – or, at least, a function of a power. As Hegel writes,

> by the exponent we are to understand here nothing but the qualitative moment itself that specifies the quantum as such. As we saw earlier, the strictly immanent qualitative moment of the quantum is solely the *power-determination*. It must be such a determination which constitutes the relation and which here, as the intrinsic [*an sich seiend*] determination, confronts the quantum as externally constituted.
>
> —SL 293 / LS 377[29]

The exponent we are considering sets a specific limit to the way in which the thing to which it belongs can be changed through the addition of a given quantum. Yet it does so as a power-determination, not as a simple, fixed quantum or number. Accordingly, the amount added to the thing and the

increase in the thing itself do not stand in an immediate, *direct* relation to one another. They obviously do not stand in an inverse ratio, since both increase (or decrease) together; but nor can they stand in a direct ratio, since they are not both simple quanta whose ratio can be fixed as 1 : 2, or 1 : 4, and so on. As the amount added to something increases, Hegel states, the thing itself undergoes a series of increases that are "numerically incommensurable" with the increases in that amount, since the increases in the thing are determined by a power-determination. This is not to deny that the relation between a measure and the quantum it specifies is "constant" (*konstant*). Yet such constancy is not grounded in a fixed quantum; it is grounded in the distinctive *quality* of the exponent that is specific to a thing – in the distinctive character of its power-determination (SL 292-3 / LS 376-7).[30]

In the remark following 1.3.1.B.b Hegel provides a concrete example of the specifying measure, namely the way something changes in *temperature* as heat is added to it (though, as we shall see, such change is in fact more complex than the pure measure itself). Assume that there are bodies in a "general medium", say, the air. As the temperature of the air increases in a uniform manner, Hegel maintains, "it is assimilated differently by the different particular bodies found in it, since they determine the externally received temperature through their immanent measure" (SL 293 / LS 377). Hegel goes on to point out that the differences in the ways in which bodies absorb heat can be quantified: the exponent governing the relation between the heat transferred to them and the increase in their temperature can be expressed as a quantum or number. This quantum constitutes the "specific heat capacity", or more simply the "specific heat" (*spezifische Wärme*), of the respective body.

Note that Hegel does not equate specific heat directly with a power-determination (as one might expect, given the logical structure described above). He simply states that it has a certain numerical value at a given temperature that allows the specificity of different bodies under similar conditions to be compared quantitatively. As he puts it, "different bodies compared at one and the same temperature give the ratio-numbers [*Verhältniszahlen*] of their specific heats, of their thermal capacities" (SL 293 / LS 378). Furthermore, in the main text of 1.3.1.B.b, he indicates how those numbers can be determined, how the specific heat of a body can be calculated. He writes that "the exponent which constitutes the moment of specificity [*das Spezifische*] can at first seem to be a fixed quantum, as *quotient* of the ratio between the external and the qualitatively determined quantum" (SL 293 / LS 377, emphasis added). If we take the exponent mentioned here to be a body's specific heat, we can see that Hegel has accurately stated how the latter is to be found. The quantitative relation between the heat transferred to a body from outside (Q) and the change in the temperature of the body itself (T) can be expressed as follows: $Q = m \times c \times \ T$ (where m is the mass of the body and c

is its specific heat).[31] If we then take two bodies of the same mass and subject to the same heat transfer, but with different specific heats, we can calculate the increase in temperature in each as follows: $\Delta T = \frac{Q}{m \times c}$. Similarly, if we know the heat transferred, the mass of each body and the increase in its temperature, we can calculate its specific heat as follows: $c = \frac{Q}{m \times \Delta T}$. If we assume in this case that the mass of both bodies is the same – 1 kg – then the specific heat of each body, or its "exponent", is the *quotient* produced by dividing the heat transferred to the body (the "external" quantum) by the increase in the temperature of the body (the "qualitatively determined quantum"), just as Hegel describes.

This quotient or exponent will be a quantum or number. Yet in the lines just quoted Hegel states that the exponent "can at first *seem* to be a fixed quantum", thereby suggesting that it is in fact not merely such a quantum. In the purely *logical* form of the specifying measure, the exponent is not merely a fixed quantum because it is a power-determination (e.g. x^2 or x^3). As noted above, however, Hegel does not equate specific heat directly with a power-determination, so how can it not just be a fixed quantum? It is not just fixed, Hegel maintains, because "the thermal capacities" – or specific heats – "of bodies *vary* [*ändern sich*] in different temperatures" (SL 293 / LS 378). The specific heat of a body will always be some *definite* quantum or number at a given temperature, and so in that sense will be "fixed"; but overall it will not constitute a "fixed ratio exponent" because *specific heat changes with temperature*.[32]

The fact that specific heat changes with temperature introduces a complexity that is not present in the purely logical structure of the specifying measure. In the latter, there is just *one* measure that specifies the quantum added to it – a measure that is in some sense a "power-determination". In Hegel's concrete example, however, the specifying measure, or specific heat, of a body – which in itself is a simple "ratio-number" – depends on the temperature of the surrounding medium (and thereby also of the body concerned). Moreover, this temperature is not simply an "external" *quantum*, but is "the temperature of the air or some other specific temperature" – the temperature of something with an explicit and distinctive *quality* – and, as such, is a *specifying measure* in its own right (SL 294 / LS 378). There are, therefore, two specifying measures in relation to one another: a body's specific heat determines the way it is affected by the outside temperature and by changes to the latter (that is, by the heat transferred to the body), but such changes in temperature in turn specify and determine the way in which a body reacts to the heat transferred to it.[33]

Following the logic of measure, however, temperature – as a specifying measure – should determine heat capacity "in accordance with [*nach*] a power-determination" (SL 293 / LS 377). This suggests in turn that the specific heat of a body at a given temperature should be proportional to, or a function of, a

certain power *of* the temperature. This should be the case at least, if the temperature is to play the role of a properly specifying measure; and in this case, the body's specific heat itself will be a properly specifying measure, since it will be a function of a power after all. Of course, however it is determined, the specific heat of a thing at a given temperature specifies the effect that heat transfer has on it; but such specific heat explicitly realizes the "concept" of a specifying measure insofar as it is a function of a power.[34]

By the way, the fact that specific heat varies with temperature does not mean that it is not, after all, the "constant" exponent of the relation between the heat transferred to a body and the increase in the latter's temperature. It is, indeed, constant; it is such, however, not only because it has a single numerical value at a given temperature, but also because it passes through a series of values that are characteristic of, and *specific to*, a body (or the material of which it is composed).

Hegel's remarks on specific heat were made in the first decades of the nineteenth century, but they anticipate insights that were developed at the start of the twentieth century by Albert Einstein and Peter Debye. In 1819 – in Hegel's own lifetime – the French physicists, Pierre Louis Dulong and Alexis Thérèse Petit, formulated their famous law stating that the specific heat capacity of a body is (very nearly) proportional to its atomic weight and so is a fixed magnitude, which in turn implies that it is independent of temperature.[35] In 1907, however, Einstein "developed a general theory of the *variation* of Cv with T for all solids at all temperatures, even down to 0 K" (where Cv is the heat capacity measured at a constant volume and T is the temperature).[36] A few years later, Debye then proved that in solids "heat capacity decreases *as* T^3 at low temperatures, in agreement with experimental observation" – and also with the speculative logic of measure.[37] At higher temperatures the specific heat of solids is indeed relatively stable within a certain range (in accordance with Dulong and Petit's law); yet at low temperatures their specific heat realizes the concept of a specifying measure in a Hegelian sense by being proportional to, and so a function of, a *power*, namely the cube of the temperature.[38] It is beyond my competence to say any more about heat capacity. It is, however, remarkable that Hegel's speculative logic should accord, explicitly and implicitly, with insights that later became central to modern physics.[39]

THE REALIZED MEASURE: GALILEO AND KEPLER

Let us recapitulate. Measure is first understood to stand alone as "some determinate quantum" to which "some determinate quality" is immediately attached (SL 288 / LS 371). As such it is "the *immediate* unity of the qualitative and the quantitative" (SL 285 / LS 368). It then proves to be a measure in *relation* to a quantum that lies outside it. As such, it is initially just a standard

(*Maßstab*) in terms of which that external quantum is measured. This, however, reduces the measure to little more than a quantum itself: it is the quantum that provides the unit, of which the other quantum is an amount (SL 291-2 / LS 375). Yet a measure is precisely a *measure* and not merely a quantum; in its first incarnation – before it becomes a standard – it thus sets a limit to the changes that can be undergone by its own quantum and in that sense negates the latter. When measure in relation to an external quantum becomes an explicit measure in this sense, it becomes a *specifying* measure. This measure negates the external quantum by limiting, and so changing, the way the latter changes it. It brings its own specific quantum to bear on the magnitude that is added to it by the external quantum and thereby specifies that magnitude. In so doing, the measure also asserts its *qualitative* character against the external quantum. The measure's specific quantum is thus not just a simple fixed quantum but a "*power-determination*" (or a function of a power) (SL 293 / LS 377).

This logical structure is exemplified by the specific heat of a body, through which the body modifies the heat transferred to it from outside (though the specific heat of solids is clearly a function of a power only at low temperatures). The third form of measure is thus not just a form of thought, but underlies a significant phenomenon in nature and explains why, say, a metal absorbs heat at a different rate from that of water. In Hegel's example, however, there are actually *two* specifying measures: the specific heat of the body and the outside temperature that itself specifies that specific heat. The example thus contains more than the logical structure it exemplifies and so more, strictly speaking, than has been shown to be necessary at this stage.

At the start of 1.3.1.B.c, however, Hegel brings us back to that simpler logical structure in which there is just one specifying measure in relation to an external quantum:

> The *qualitative* side of the quantum, determinate in itself, exists only as a relation [*als Beziehung*] to the externally *quantitative* side; as specifying the latter it is a sublating of its externality through which the quantum as such is. This qualitative side thus has a quantum for its presupposition and its starting point.
>
> —SL 294 / LS 378, emphasis added

Hegel then proceeds to derive a new form of measure from the structure he has just described by rendering explicit what is implicit in the latter.

Hegel points out first that the merely external quantum we have been considering is not in fact purely quantitative after all, because it is itself "qualitatively" – and so immediately – different from quality, that is, from the qualitative, specifying measure (SL 294 / LS 378). This in turn reflects the fact that quantity *as such* is qualitatively – and so immediately – different from

quality: it is a further form of quality that no longer exhibits the characteristic logical structure of quality itself.[40] For this very reason, however, the external quantum in the specifying measure is explicitly *quantitative*, not qualitative; this is why it is subject to specification by the measure and not the other way around. The quantum, as quantum, is thus only *implicitly* qualitative, and in that sense there is only an implicitly qualitative (or immediate) difference between it and its specifying counterpart: the difference is not, explicitly, between one immediate quality and another. When, however, in accordance with speculative method,[41] that implicitly qualitative difference is rendered explicit – or, as Hegel puts it, is "posited in the *immediacy* of being" (SL 294 / LS 378-379) – both sides in the relation have to be conceived as explicitly qualitative. This means in turn that the quantum on each side is not just a quantum but the specific quantum *of a quality*.

This takes us to a new logical structure that must be carefully distinguished from its predecessors. Both the rule and the specifying measure confront a quantum that is, or belongs to, something other than the measure (see 2: 250-1). Such a something in turn necessarily has a certain quality; indeed, in the case of the specifying measure that something must have (in some respect) "the same quality" as the measure itself (which enables the former to act on the latter and the latter to specify the effect the former has on it) (SL 292 / LS 376).[42] Yet in these two cases, the external quantum specified by the measure is a matter of *indifference* to the quality of the thing with that quantum; it is not explicitly the thing's measure and so in that sense is not explicitly "qualitative". The quantum belongs to something *with* a quality, and implicitly constitutes its measure since it permits the thing to be what it is; yet it is explicitly a mere "measureless" (*maßlos*) *quantum* – the mere quantum that the rule and the specifying measure require as their logical counterpart (SL 292 / LS 376).

In the new logical structure, by contrast, that quantum is now itself explicitly qualitative. This means not just that it belongs to something *with* a quality, but that it is explicitly specific *to* this quality. The quantum that is external to the specifying measure is now no longer just a quantum but the "quantum *of* a something and *of* its quality" (SL 294 / LS 379, emphasis added). Both sides of the relation, therefore, now have the same logical structure: each is explicitly quantitative *and* qualitative.

Note, however, that this shared structure does not eliminate the difference between the two sides. The reason why is that the external quantum becomes qualitative when we render explicit the implicit *qualitative difference* between it and its specifying counterpart. As Hegel puts it, it is "this difference between the two" that is posited in the "*immediacy* of being" (SL 294 / LS 378-379). So, although the external quantum does, indeed, become qualitative, like its counterpart, it does so as it becomes explicitly *different* qualitatively from the latter. The two sides in the new logical structure must, therefore, have their

own distinctive qualities, and the quantum that each is must be the specific quantum of *that* quality. It is, of course, possible, as a matter of fact, to encounter two related things with the same specific quantum and same quality, such as two equal amounts of water; but such a relation between things is not what is made necessary at this point by the logic of measure. What is made necessary here is a relation between two things, each of which has its specific quantum and the distinctive quality associated with the latter. In the new measure, therefore, two *quanta* now coincide with two different *qualities* in relation to one another.[43]

There is, however, a subtle logical difference between the things *as* qualitative and *as* quantitative. As qualitative, they are immediately distinct from one another, since they are two different somethings: as Hegel puts it, "each is for itself [*für sich*] such a determinate being" (SL 294 / LS 379).[44] As such, therefore, they are not explicitly related *to* one another: they are not connected by their different qualities, but are simply something and something else.[45] In the previous "specifying" measure, however, measure took the form of the explicit *relation* between two *quanta* (in which one altered or "specified" the other). This remains the case in the new measure, since the latter simply renders explicit what is implicit in its predecessor. Accordingly, although the two things in this measure are, as qualitative, not explicitly related, they *are* explicitly related to one another by their specific magnitudes. As Hegel puts it, "measure is thus the *immanent* quantitative relating of *two* qualities to each other" (SL 295 / LS 379).

This measure is, more precisely, a *single* measure – "*ein* Maß" (SL 291 / LS 375) – that consists *in* a quantitative relation between qualities. Moreover, since each side of this relation is the "specific magnitude" of a quality, each is itself a measure in its own right (SL 294 / LS 379). The relation between them is thus in fact one of *measure to measure*, and so constitutes measure that is *for itself*. Accordingly, when Hegel writes that the two qualities are "implicitly connected to each other in the *being-for-self* of measure", he is pointing out that they are related to one another, not just through their magnitudes, but through their *measures*. The new measure we are considering is *one* measure that is a relation *between* two measures.[46]

Now, as we know, a measure as such contains a quantum in two senses: as constituting the measure itself – as the quantum that is specific to, and sustains, the quality of the thing concerned – and as an immediate, "external" quantum that can change (and exceed the measure of the thing) (2: 248-9). Accordingly, the two quanta in the new measure must also be both kinds of quantum. In Hegel's words, "the quantum is in its double being [*Doppelsein*] external and specific, so that each of the distinct quantities has this twofold determination in it and is at the same time inextricably bound to the other" (SL 294 / LS 379). Each, therefore, must be a merely external, changeable quantum in relation to

another such quantum, but each must also be a specific quantum that belongs specifically to this quality rather than that.

Hegel argues that this requires the new measure to take *three* different forms, depending on which aspect of the measure is more to the fore. Two of these will mirror measures we have already encountered, whereas the third will be unique to this new measure and, indeed, will alone be the full realization of the latter. As we shall now see, there must be multiple forms of this new measure, because initially the latter will just relate simple immediate quanta to one another, not quanta that are *explicitly qualitative as quanta*, that is, quanta raised to a power. The first form of the measure will thus not fully realize the logical structure, or "concept", we have just set out. It will only be in the third form of the measure that its logical structure will be fully realized and we will encounter two specific magnitudes, in the form of powers, specifying and determining one another.

As we have noted, each quality in the new measure has a quantum that belongs specifically to it (and so each has its own measure). In the initial, immediate form of the new measure, however, each constituent measure must itself be some *immediate* quantum that is attached to its quality: as Hegel puts it, the two sides in the relation are "taken at first simply as determinacies of magnitude [*Größebestimmtheiten*]" (SL 295 / LS 380). The new measure thus consists first in the relation between these simple magnitudes. It is a definite, fixed relation between them, because it has a determinate character of its own that makes it the measure it is; yet the two quanta in the relation, as simple, immediate *quanta*, are also inherently *changeable*. The distinctive "determination of the measure" (*Maßbestimmung*) thus resides in a fixed relation, or *direct ratio*, between two changeable quanta: so, as one increases, the measure requires the other to increase by a proportional amount.[47] As an example of this measure, Hegel points to velocity, in which a certain quantum of space is traversed in a given time: say, two metres per second. The distance travelled can increase from two to four metres, but the measure is preserved insofar as the direct ratio between distance and time – the velocity – is preserved: so four metres are traversed in two seconds (SL 295-7 / LS 380-2).[48] Velocity might seem to be a purely arbitrary relation between distance and time, but in fact it combines the two aspects of measure noted above. The magnitudes of the distance and time are, indeed, simply given, and so arbitrary, and the velocity could just as easily have been another. Yet the velocity is the *measure* of a certain (uniform rectilinear) motion that gives it its distinctive character, and each magnitude, as a moment of that measure, is *specific* to its quality and stands in a fixed relation to its counterpart (even as it changes its numerical value).[49]

The two sides of the new measure must, however, be more clearly differentiated from one another than this, since their relation must also render explicit the *difference* within measure between measure itself and the mere

quantum. This difference is already present in velocity, since, as in the rule, one of the qualities provides the "unit" through which the other's amount is "specifically determined" (SL 295 / LS 380): velocity is distance-per-unit-of-time (or time-per-unit-of-distance). Just as in the rule, however, the unit can itself be regarded as an amount, so each side remains a given quantum. To understand how measure and quantum can be *explicitly* distinguished in the new measure, therefore, we must look back to the measure that comes after the rule. In this measure, one moment remains a mere quantum, but the other is the "specifying" *measure* that explicitly negates and changes the first (rather than just subordinating it to a rule). It does so by asserting its distinctive *quality* – or, in Hegel's words, "the qualitative moment" – against its counterpart (SL 293 / LS 377).

Now, as we know, a measure as such is a quantum that is specific to, and so one with, a quality. The unity of quantum and quality can, however, be more or less explicit. In the first, *immediate* measure (described in 1.3.1.A), the quantum to which a quality is attached is itself simply immediate: it is just "some determinate quantum" (SL 288 / LS 371). Yet at the end of the account of quantity the quantum proves to be explicitly qualitative by raising itself to a *power* of itself: for, in doing the latter, it relates solely to *itself* in becoming another quantum and thereby exhibits the quality of "being-for-self" (SL 278 / LS 359). It follows that a quantum is most explicitly united with quality as a *measure*, when it is not just "some determinate quantum" but one raised to a power. Logically, therefore, the specifying measure (in 1.3.1.B.b) that explicitly differentiates itself from and acts on the mere quantum must change the latter "in accordance with a power-determination" (SL 293 / LS 377) (or, in the case of specific heat, a function of a power).

Since the two quanta in the new measure (in 1.3.1.B.c) must also be distinguished from (and related to) one another as explicit measure and mere quantum, the former must also determine the latter by raising itself to a power of itself. This yields the second form of the new measure. Hegel's example is Galileo's law of falling bodies, according to which distance is proportional to the *square* of the time, or (as Hegel expresses it) $s = at^2$ (SL 296 / LS 381). The distance and the time needed to traverse it are both changeable quanta, but, as in the case of simple velocity, their relation to one another is fixed. In this case, however, to calculate the distance travelled in an increased time – say, in three seconds, rather than one – the initial distance-per-second is multiplied not just by the new time, but by the square of the new time. So, as Hegel explains in an addition to his *Encyclopaedia Philosophy of Nature*, if "the body falls a little more than 15 feet in the first second", "in two seconds, the body falls, not twice but four times the distance, i.e. 60 feet; in three seconds it falls 9×15 feet, and so on" (EN 60 / 79 [§ 267 A]).[50] This relation between distance and the square of the time does not characterize all movement, but it is the distinctive measure

of freely falling, and thereby uniformly accelerating, bodies (on a planet or moon); and Hegel claims that it is logically necessary that there be a measure with this form.

The new measure that has arisen at this point in the *Logic* thus takes the form of two distinct relations between quanta. Yet Hegel argues that it also takes the form of a third relation, in which, indeed, this new measure is most fully realized. This third relation renders explicit the fact that the measure is a relation between *two measures*. The two sides are thus not just quanta, and are not just related as quantum and its specifying counterpart, but both are quanta raised to a power – though each is raised to a different power, and so "specified" in a different way. The example Hegel gives of this third form of the new measure is Kepler's third law of planetary motion: the principle that the squares of the orbital periods of any two planets are proportional to the cubes of their mean distances from the sun, or (in Hegel's expression) $s^3 = at^2$ (SL 296 / LS 381).[51] Once again, not all motion is subject to this law, but the law is a specific measure of planetary motion, and it exemplifies a form of measure that, in Hegel's view, is made necessary logically by the nature of measure itself.

Hegel calls the measure we have been considering in this section "the *realized measure*" (SL 295 / LS 380). This measure realizes itself most fully, however, only in the third of the three forms it takes: for only in this case are the two measures related to one another *explicitly* as quanta that are "qualitatively determined" and so as *measures*. The "higher realization of the qualification of the quantitative", Hegel states, "is this: that both sides are related in higher determinations of powers (as in $s^3 = at^2$)" (SL 296 / LS 381).[52] (As we will see in the next section, the qualified quanta, or power-determinations, that are moments of the second and third forms of the realized measure, are also simple quanta with a numerical value. Quanta are thus explicitly present in those forms in their "double being", as the concept of the measure requires [SL 294 / LS 379].)

QUALITY AND QUANTITY IN THE REALIZED MEASURE

The first part of Hegel's account of the realized measure is now almost complete; there is just one more thing to be explained. This measure, as we have seen, takes the form of a relation between qualities, which, however, are related by the *quanta* or magnitudes that are specific to them. It is, therefore, a "quantitative relating" of qualities to one another (SL 295 / LS 379). Indeed, the quanta actually stand in a threefold relation: (i) as one simple quantum related to another such quantum, (ii) as one simple quantum related to a power, and (iii) as one power related to another power. These three relations reflect the fact that a quantum, as a measure, is a simple, external *quantum*, but also a quantum

that is one with *quality*. So, a quantum as measure is initially (in 1.3.1.A) just an immediate *amount* that sustains a quality; but in its more developed form (in 1.3.1.B.b) it is a quantum that has been "specified", "qualified" and turned into a *power* by that quality (and, as such, specifies another quantum). Moreover, as a power, a measure unites both kinds of quantum, since it is precisely the power *of* some *quantum*.

The quantitative aspect of each side of the three relations in the realized measure is thus determined by what it is to be a quantum in and as a measure. As yet, however, the *qualitative* aspect of each side is not determined in the same way, and so remains independent of the quantitative aspect. This independence, however, cannot be allowed to stand, for in a measure quality and quantity are one.

The independence of the qualities from the quanta in the relation can be seen in two features of those qualities. First, since those qualities are immediately distinct from one another (see 2: 260), they are simply and immediately *given*; or, as Hegel puts it, they are just "qualities as such" with "some particular meaning or other, for instance, space and time" (SL 295 / LS 380). Their character or "meaning" does not, therefore, derive from the nature of the quanta in the relation. This, however, is at odds with the idea that the quanta are specific quanta, or measures, and so in some sense *constitutive* of quality. Second, nothing in the qualities themselves determines which one should be the unit to the other's amount (in the simple relation between quantum and quantum) or which one should be the power (in the relation between quantum and power). It remains unclear, therefore, why $s = at^2$, rather than $t = as^2$. Insofar as each is "only a particular quality", Hegel writes, there is "no way of distinguishing which of them is to be taken with respect to their determination of magnitude as merely externally quantitative and which as varying in quantitative specification". They are thus "indeterminately different" (*unbestimmt verschieden*) (SL 296 / LS 380-1).

Hegel now explains, however, that the two qualities cannot just be indeterminately different, "for as moments of measure the qualification of the latter should reside in them" (SL 296 / LS 381). That is to say, since quality is united with quantity in measure, the two qualities in the realized measure should differ in a way that coincides with the quantitative difference in the measure. Indeed, the two qualities must be determined by the two ways in which the quantum is present in the measure.[53] The quantum is present, as we have seen, as a mere quantum and as a power (that is, as qualitative). It is this difference that must ground the difference between the two qualities.

Accordingly, one of the qualities must reflect the sheer externality of the simple quantum, and so will be what Hegel calls "the *extensive*". The other, by contrast, must reflect the unity and self-relation of the qualitative quantum, and so be what Hegel calls "the *intensive*, the being-within-self or negative relative

to the other" (SL 296 / LS 381). Note that the terms "extensive" and "intensive" apply here, not, as earlier, to two types of quantum (see SL 182-3 / LS 231-2), but to two different *qualities* that reflect the different forms of the quantum in a realized measure. This qualitative difference in turn explains why, in the first form of the realized measure, the quantum of one quality is the amount and the quantum of the other is the unit: the qualitatively "extensive" moment – space, in Hegel's example – will provide the amount, and the "intensive" moment – time – will provide the unit (so velocity is normally conceived as distance per unit *of time*) (see SL 296-7 / LS 381-2).[54] In this sense, therefore, the two qualities in the realized measure, which reflect the different forms of the quantum in it, are *themselves* the ground of the quanta that attach to them (even though the actual numbers of the amount and unit – e.g. two metres per second – remain simply given and in that respect logically ungrounded).

The qualitative distinction between being extensive and being intensive is also meant to explain why, in the second form of the current measure, the quantum of one of the qualities is raised to the power of itself, and so squared, whereas that of the other is not. The details of this explanation, however, are not provided in the *Logic*, and the brief remarks that Hegel does make initially appear contradictory.[55] He maintains first that in the "specifying relation", as exemplified by Galileo's law of fall, the *extensive* quality – space – is to be taken "as the power or the becoming-other" since it extends beyond itself, and the intensive quality – time – is to be taken "as the root" (SL 296 / LS 381). Yet time, as intensive, provides the unit that determines – or "specifies" – the distance travelled by a falling body; so that distance, or "space", is in fact proportional to the square *of the time*: $s = at^2$. The appearance of contradiction is, however, removed in the following lines: for there Hegel notes that, although the time is squared, it is the magnitude of *space* that is determined or "specified" *by* that squaring (and so in that sense space is "taken as the power"). As the time increases arithmetically, therefore, the *distance* travelled by a freely falling body in the longer time is the initial distance-per-time-unit multiplied by the square of the new time: so, as noted above, if in 1 second the body falls 15 feet, in 2 seconds it will fall, not 30 feet, but 15×2^2, or 60, feet.[56]

In this second form of realized measure, the fact that there is a distinction between a simple quantum and a power at all is determined by the two ways in which a quantum appears in measure as such. It is the distinct character of the qualities, however, that determines which one is the power and which not. In that sense, the second form of realized measure is grounded in, and specific to, *these qualities*. The same is true of the third form of realized measure: the qualitative difference between "intensive" time and "extensive" space explains (or is meant to explain) why, in Kepler's third law, the *squares* of the orbital periods of any two planets are proportional to the *cubes* of their mean distances from the sun. In the *Logic*, however, Hegel presents almost nothing of that

explanation itself; we have to wait until the *Philosophy of Nature* to learn what exactly it is about "extensive" space and "intensive" time that requires planets to obey Kepler's law.[57] What we learn in the *Logic* is simply *that* (in Hegel's words) "fundamental relations of this kind rest on the nature of the related qualities of space and time and on the kind of relation in which they stand, whether motion is mechanical [. . .] or the absolutely free motion of the heavens" (SL 297 / LS 382).

So, to recapitulate: in the realized measure, two qualities are related to one another by their *quanta*; that is, a certain magnitude of one is proportional to a certain magnitude of the other. The necessary difference between the two kinds of quanta in a measure then generates three different quantitative relations – between amount and unit, quantum and power, and power and power – that connect the different qualities with one another. The *qualities* themselves, however, determine (in the second relation) which one is the power and which a simple quantum, and (in the third relation) which is which power. In the third form of realized measure, indeed, we witness the most fully developed "qualification of the quantitative" (SL 296 / LS 381): both quanta are *themselves* qualitative in being a power and each is given its particular power *by* (and so is fully specific to) the quality to which it belongs. Yet the difference between these qualities – the "extensive" and the "intensive" – is itself grounded in the difference between the two kinds of quantum in a realized measure. In that sense, *quantity* remains the governing category in this measure.[58]

This concludes the first part of Hegel's discussion of the realized measure, the measure that consists in a quantitative relation between qualities. In the remark following that initial discussion Hegel notes once again that realized measures are to be found in nature in the form of laws, namely Galileo's law of fall and Kepler's third law. Neither of these laws is a universal law governing everything, but each expresses a relation between quanta that is *specific* to a certain phenomenon. Each, therefore, is the distinctive measure of that phenomenon. Hegel goes on to maintain that "the *mathematics of nature*, if it wants to be worthy of the name of science, must be essentially the science of measures" (SL 297 / LS 382). There is great merit, he admits, in discovering "the empirical numbers of nature", such as the distance of the planets from one another; but science's more significant role lies in understanding quantities in nature as "moments of a *law* or measure". It is for this reason that Hegel bestows such praise on Galileo and Kepler (and would presumably have welcomed Einstein's formula, $e = mc^2$).[59]

The laws of Galileo and Kepler were initially "*proven*" (*erwiesen*) by being shown to correspond "to the details of perception", to render the empirical world intelligible. Yet Hegel also thinks that "a higher *proof* [*Beweisen*] of these laws must be demanded". Such proof cannot be provided by mathematics alone, he claims, but must involve "knowing their quantitative determinations from the

qualities or determinate concepts connected in them (such as space and time)" (SL 297 / LS 383). As noted above, this proof is provided in part by speculative logic and then more fully by the philosophy of nature. Philosophy thus incorporates, and is indebted to, the independent discoveries of natural science, but it then provides a purely a priori proof of the laws that have been discovered to complement the empirical proof that they have already been given.[60]

BEING-FOR-SELF IN MEASURE: THE "EMPIRICAL COEFFICIENT"

It can be tempting to admire the consistency of Hegel's derivation of categories in the *Logic*, but to wonder what the point of it all is. In Hegel's view, however, the point is one of great significance. Logic shows not just that certain categories are conceivable, but that they – and the corresponding ways of being – are logically *necessary*. It does so by demonstrating that they are inherent in thought – and being – itself; and it does this by rendering explicit what is implicit in the category of pure being and the subsequent categories that arise. The logic of measure thus shows that certain forms of measure belong of necessity to the very fabric of being. The occurrence of these forms of measure in *nature* is then confirmed by the examples that Hegel provides.[61]

Hegel's logic of measure is thus not just a "reconstruction" of concepts from the history of philosophy or science; nor is it just a critique of inadequate ways of thinking "determinately" about reality.[62] It is a positive metaphysics that discloses the measures there must be in the world. This in turn means that, for speculative philosophy, certain forms of motion and the laws that govern them are not just contingent, but exemplify being's very own measures. This, however, is not to deny that there is contingency in the world. Indeed, Hegel argues that such contingency is actually an integral feature of measure itself.

It has been noted above that in a measure as such the quantum has a "double being" (SL 294 / LS 379): it is, on the one hand, a mere quantum and, on the other, a quantum that is specific to a certain quality (and thus the measure of the thing concerned). This double being is evident in the fact that the very first measure (in 1.3.1.A) is a quantum that sets a limit to the changes it can undergo as a mere quantum; and the duality becomes explicit in the relation between, first, the rule and the quantum and, second, the specifying measure and the quantum. As we have just seen, this double being also manifests itself in the "realized" measure by requiring the latter to take three forms: the relation between two quanta, the relation between a quantum and a "specifying" quantum in the form of a power-determination, and the relation between two power-determinations. The double being, however, also manifests itself in the fact that the last two relations themselves coincide with relations between

simple, given quanta; and it is this fact that places *contingency* at the heart of the realized measure. We shall now consider why this is the case.[63]

A simple quantum is by its nature contingent, since it is simply and immediately what it is and could just as well be different; there is thus contingency in the fact that there is *this* much water, rather than that much, in the sea. There is also contingency in the fact that something has its measure in *this* immediate quantum – that water boils at 100° C, rather than 60° C – and the fact that the first form of realized measure, velocity, has *this* magnitude. These measures, as *measures*, are logically necessary; but they necessarily contain contingency, because the quanta they involve must be immediately given and thus be beyond explanation by logic. This is not to deny that natural science might be able to explain why water boils at 100° C or an object travels at a certain speed, but logic alone cannot do so.[64]

Now logic, as we have seen, requires the realized measure to take two other forms beyond that exemplified by velocity. In these forms, however, the quanta concerned are not just immediate, and so contingent, but one has, or both have, a character that is further "specified" by the logic of measure. In the Galilean realized measure one quantum is thus a power-determination, and in the Keplerian measure both quanta are. The Keplerian measure in particular is, therefore, explicitly the relation between two "qualitative" quanta, or *measures*, rather than between two merely immediate and contingent quanta.

Yet a measure as such is both a measure and an immediate quantum *at the same time*. This should be the case, therefore, in both the last two forms of realized measure. Moreover, there is a further reason, more specific to the realized measure itself, why the two sides of all three of its forms must be immediate quanta. As we saw earlier (2: 260), the qualities in this measure are immediately distinct from one another. They are related to one another by their specific magnitudes, but as qualities they are not explicitly related: as Hegel puts it, "each is for itself such a determinate being" (SL 294 / LS 379). In the *Philosophy of Nature*, Hegel will show that space and time are connected in themselves, since time is the self-negating *of* space – its never-ending disappearance.[65] Such an intrinsic connection between qualities has not, however, been shown to be necessary to the realized measure. Consequently, the qualities are bound together in that measure *only by their quanta*, and are otherwise quite separate from one another. In Hegel's words,

> the qualities are still posited only as immediate, *only as diverse* [*verschiedene*]; they are not related in the manner of their quantitative determinacies, that is, such that *outside* their relation they would have neither meaning nor existence, as is the case with the quantitative determinacies as powers [*Potenzenbestimmtheit der Größe*].

Accordingly, in the realized measure,

> space and time, outside that specification which their quantitative determinacy obtains in the motion of falling bodies or in the absolutely free motion, both have the value of space as such and time as such, space subsisting on its own outside and without the duration of time, and time flowing on its own independently of space.
>
> —SL 298 / LS 384

Yet these immediately given and immediately distinct qualities are moments of a measure and so are inseparable from quantity. Such quantity in turn must, therefore, take *two* forms. On the one hand, it is the threefold quantitative relation that we have been examining. On the other hand, however, it must also take the form of an *immediate* quantum attached to each quality in *its* immediacy. The qualities of "extensive" space (s) and "intensive" time (t) in the realized measure are thus related to one another in the following ways: $v = \frac{s}{t}$, $s = at^2$, and $s^3 = at^2$; but equally each "immediate quality also has only an *immediate quantum*" (SL 299 / LS 385). S and t must therefore always have some *given* (and in that sense, immediate) numerical value. Velocity, for example, is never just velocity in the abstract, but always 5 miles an hour, or 2 feet per second, and so on; and this is not just an empirical fact about velocity, but it is made necessary by the logical structure of the realized measure.

Indeed, the immediacy of the immediate quantum in a realized measure is even more thoroughgoing than that of the quality to which it belongs. Each quality is immediate insofar as it lacks an intrinsic connection to its counterpart and so is just itself. Yet, as we saw above, these qualities are not *purely* immediate, since the difference between them must reflect the different ways in which a quantum is, and must be, present in a realized measure; hence one quality must be extensive (exemplified by space) and the other intensive (exemplified by time) (2: 265). By contrast, the immediate quantum that necessarily attaches to each quality is purely immediate, since there is nothing in the nature of the qualities themselves, nor anything in the measure to which they belong, that requires them to have *this* quantum and *this* amount, rather than any other. As immediate, therefore, the quantum of each quality is simply the particular, given – "empirical" – magnitude that the quality happens to have.

Hegel stresses that, at this point in the *Logic*, "immediacy is a moment of those [sides] which themselves *belong to measure*" (SL 299 / LS 385, emphasis added). The immediate quality and its immediate quantum are not free-floating entities but aspects *of* the two sides of the realized measure: the immediate quanta are the given numerical values of *s* and *t* in $v = \frac{s}{t}$, $s = at^2$, and $s^3 = at^2$. Furthermore, since the measure, in each of its three forms, is essentially a *relation* between its constituent elements, it must also bring these immediate qualities

and quanta into relation to one another. Accordingly, Hegel writes, "the immediate qualities also belong to measure, are likewise in relation [*in Beziehung*] and stand, according to the determinacy of magnitude, in a ratio [*Verhältnis*]" (SL 299 / LS 385). This ratio, he notes, must be a *direct* ratio.[66]

We saw in the section on quantity that simple immediate quanta can stand in a direct or an inverse ratio to one another. The immediate quanta in a realized measure cannot stand in an inverse ratio because the two sides of such a measure increase or decrease together; this in turn follows from the fact that the sides in the specifying measure, from which the realized measure emerges logically, also increase and decrease together (though, of course, in that case the one alters the way the other alters it). The quanta in a realized measure must, therefore, stand in a direct ratio to one another. This measure has thus proven to be somewhat more complex than we thought. It is (in its two higher forms) a relation involving (one or two) qualified or "*specified*" quanta of qualities, or powers, that at the same time is, or contains, a relation between simple, *immediate* quanta of those same qualities – "a ratio which, outside the specified one of power determination, is itself only a direct ratio and an immediate measure" (SL 299 / LS 385). The first form of the realized measure already consists in a direct ratio: velocity, for example, is simply the ratio between distance and a unit of time. Hegel has now shown that the second and third forms of the realized measure, which involve powers, also entail direct ratios between immediate quanta.

Note that the moment of immediacy is not to be found in the particular powers to which quanta are raised in these measures – whether a quantum is squared or cubed – since (as indicated above) Hegel argues that these powers are themselves necessary.[67] Immediacy is to be found, rather, in the quanta that are raised *to* a power, namely the quanta of s and t (distance and time). Each power in the laws that exemplify the second and third realized measures is thus the power *of* an immediate quantum.[68] This means in turn that any power of s and t must also be an immediate quantum: $6^2 = 36$, and so on. The measure as a relation of *powers* – $s = at^2$ or $s^3 = at^2$ – must, therefore, *itself* be a direct ratio between such quanta, and this ratio must have an exponent that is found by dividing one side of the ratio by the other. Accordingly, as Hegel writes, the ineliminable element of contingent immediacy at the heart of a realized measure is "a simple immediate quantum, the quotient or exponent of a ratio between the sides of the measure, which is taken as direct" (SL 299 / LS 385). It is what Hegel calls the "empirical coefficient" in such a measure and it is represented in his expressions of the two laws of motion by *a* (SL 300 / LS 386).[69]

Note, however, that this "coefficient" is not only a function of *s* and *t* (or their powers), but also the immediate quantitative determinacy *of the measure*: it is the quantum that gives the measure its distinctive empirical character. As such, it is the moment of fixed immediacy or "*being-for-self*" in the measure that

limits the change of s and t (SL 299-300 / LS 385-7) – though, of course, it can itself change and so give rise to a different empirical measure.[70] The change of s and t is thus limited in two ways by the two aspects of the measure. On the one hand, it is limited by the logical form of the measure: so in the Galilean measure, the distance travelled by a falling body must be proportional to the *square* of the time that has passed, or $s = t^2$. On the other hand, such change is also limited by the "empirical coefficient" in the measure: so $s = a \times t^2$ (where, for Hegel, *a* is the distance the body falls in the first unit of time).[71]

Freely falling bodies thus always fall in accordance with Galileo's law, but their rate of fall is also governed by a particular *number* that is immediate. This number is not determined by the logic of measure (or by the nature of space and time) and so, from the point of view of speculative philosophy, is *contingent*. The logic of measure does, indeed, make it necessary that there be this moment of contingency in the realized measure, but what is made necessary is precisely an immediate quantum that is simply something "externally given" (SL 299 / LS 385).[72] Logic, in other words, requires that a realized measure have an empirical coefficient that logic itself cannot determine; and since this coefficient is the moment in measure that cannot be determined by the logic of measure, Hegel describes it as the "negation of the qualitative determination of measure" (SL 299 / LS 385).

In the case of Galileo's law of fall, therefore, the fact that the distance travelled by a falling body is proportional to the square of the time does not determine whether the body should fall three feet or fifteen feet in the first second. The particular ratio of distance to initial unit of time is simply a given feature of the world to which the body belongs, and so is "an *immediate measure*, just like the size [*Maßgröße*] of human limbs, the distances and diameters of planets, etc." (SL 300 / LS 386). This is not to deny that there may be factors that explain why bodies on *this* world fall at *this* particular rate rather than another, factors that natural science can discover; but that rate is not determined by the law of fall itself and so in that sense is external to that law or measure and is simply immediate.[73]

In his philosophy of nature Hegel reminds us that, leaving aside the effects of air resistance, bodies on the same planet fall at the same rate.[74] Their rate of fall may *differ*, however, from planet to planet and moon to moon: the same body in the same initial unit of time may fall a different distance on a different planet. The ground of the distinctive immediacy that determines the way bodies fall is thus to be found in the terrestrial (or lunar) body to which they belong.[75] The ground of the law governing their fall lies, however, in the logic of measure, and ultimately the logic of being itself.

A similar empirical coefficient characterizes the movement of the planets in our solar system, a movement governed by the third form of realized measure, $s^3 = at^2$. Each planet lies at a different mean distance from the sun and has a different orbital period; but in each case the cube of the distance is proportional

to the square of the period, and in each case the exponent of the direct ratio between the two sides is the same, namely approximately twenty-five.[76] Planets belonging to a different solar system will also obey Kepler's third law (unless contingencies in the system intervene), and the ratio between the cubes of their mean distances from their sun to the squares of their orbital periods – when these are converted into simple numbers – will also have an exponent; but the numerical value of that exponent may differ from that of our solar system.

THE TRANSITION TO REAL MEASURE

At the start of 1.3.1.C.3 Hegel sets out once more the logical structure that the measure has now proven to have. It is, he writes, a "specified quantitative relation which, as quantitative [*als quantitativ*], has in it the ordinary external quantum" (SL 300 / LS 387).[77] The measure is a "specified" relation, in its developed sense, insofar it ties together two quanta, one or both of which have themselves been specified, or qualified, by being raised to a certain power. It has an "external" quantum in it because that specified relation contains, or coincides with, a direct relation between quanta, whose exponent is itself a simple given amount (such as the distance a body falls in the first period of time). This amount, Hegel notes, is not only a given quantum, but also a "determining moment of the relation as such". This is because it further determines the *specified* relation that constitutes the measure: the fall of a body is governed by the law or measure, $s = at^2$, but it is also determined by the particular numerical value of *a*.

The measure that governs the fall of a body or the orbit of a planet is thus not just one single measure, but comprises *two* relations: a specified relation between powers (or a power and a quantum) and a direct relation between amounts. These two relations are, however, *independent* of one another: the fact that a body is subject to Galileo's law does not determine how far it should fall in the first period of time, and the fact that $s^3 = at^2$ does not require *a* to have one value rather than another. In that sense, the two relations constitute two distinct measures governing falling or orbiting bodies. Yet these two distinct measures actually constitute one measure, since any specified relation between quanta of qualities is inseparable from a direct relation between quanta and the latter will always have a given exponent: that is to say, whenever $s = at^2$ or $s^3 = at^2$, *a* must have *some* particular numerical value.

This, Hegel argues, points forward logically to a new conception of measure, for it yields the "further determination" that *one* realized measure comprises *two* measures – "one as immediate and external and the other as internally specified, while the measure is the unity of the two" (SL 301 / LS 387). The realized measure is, of course, a relation between two measures from the start, insofar as each of its relata unites quantity and quality: each is a quantum

specific to a quality, either in being the immediate amount that distinguishes the quality in the relation or in being a power determined by that quality. Conceived in this way, however, the two measures are simply the two *sides* of the relation constituting the realized measure. Now, by contrast, the two measures in the latter consist in two different *relations between those sides*: one that is "specified" and one that is direct.

We have been led to this new conception of measure by the intrinsic logic of measure itself. If we look back to 1.3.1.B.c.2, we see that the simplest realized measure is the relation between immediate quanta of two qualities, as in the example of velocity: s = at. This is a specified relation between quanta, since one of them – t – serves as the unit in terms of which the other – s – is to be measured; but this specified relation is itself no more than a direct relation between s and t. The realized measure then takes two more explicitly specified forms, in which first one of the quanta, and then each quantum, is a power. As such, neither relation is a direct ratio any more. The qualities in each specified relation, however, are *immediate* and so are attached to immediate quanta: so s and t in every realized measure must have a particular given numerical value. This means that s^3 and t^2 must have some numerical value, too, and that in turn requires the two higher forms of specified relation between quanta to contain, or coincide with, a *direct* ratio between quanta after all.

Since the latter ratio has a fixed exponent that is independent of the specified relation between quanta, the two relations in the realized measure in fact constitute two *different* measures: the fall of a body is governed both by the law of fall *and* by the particular distance the body falls in the first period of time. This means, however, that the realized measure is not just a relation between different *quanta* of qualities; it is also a relation between different *measures* that are themselves different *relations* between quanta. At this point, therefore, measure proves to have a much more complex structure than has hitherto been recognized.

Hegel now examines the relation *between relations* that constitutes the realized measure and points out that they form what he calls a "negative unity", that is, a unity in which each negates the other (SL 301 / LS 388). This occurs because each, in its *independence*, undermines the independence of the other. The two in their independence from one another thus turn out to be *inseparable* moments of one realized measure. Let us see how and why this happens.

Hegel notes that "the measure contains the relation in which the magnitudes are determined and posited as different [*different gesetzt*] by the nature of the qualities" (SL 301 / LS 387). In this "specified" relation, therefore, the qualities require their quanta to be related as powers (or as power to quantum). In this sense, the measure, as a relation of qualities, gives itself its own quantitative character or determinacy, and "its determinacy, therefore, is wholly immanent and independent". That determinacy is "independent" (*selbständig*), more

specifically, because it is not determined by any given immediacy: *however* far a body falls in one unit of time, its fall will be governed by Galileo's law. And yet, this measure is not completely independent and self-determining, because it does not determine on its own its "final *fürsichseiend* determinacy", the determinacy that gives it a quite particular character. This, as we know, is determined by the exponent of the other measure in the realized measure, namely the direct ratio that it contains or with which it coincides. The self-determining of the first, specified measure is thus "negated" by the simple immediacy and independent givenness of the second measure. And yet, we learn, that "immediate measure" – the direct ratio with its given exponent – is not wholly independent of the specified measure after all: for, in Hegel's words, it has "its qualitative determinacy" in the *specified measure* ("*an jenem*") (SL 301 / LS 387).

Hegel's text at this point is by no means easy to interpret, but one way of making sense of what he says is this: the exponent of the direct ratio never ceases being a simple quantum, rather than a power, yet it is determined "qualitatively" insofar as it is determined *by* quanta that are themselves powers. This occurs explicitly in the third, Keplerian, form of the realized measure, but not in the other two forms. In this measure, the exponent is independent of the relation of powers, insofar as it is related to the given distances of the planets from the sun and their given orbital periods: in our solar system, *given* those distances (decreased by a factor of a million) and orbital periods (calculated in earth days), $s^3 = 25 \times t^2$.[78] Yet the particular exponent, 25, is produced *only* when s^3 is divided by t^2, *not* when s is divided by t. In that sense, it is not completely independent of the ratio between the powers but is partly determined by that ratio.

The Galilean measure has a purely given exponent;[79] in the Keplerian measure, by contrast, the two independent measures – the ratio of powers and the direct ratio with its exponent – *determine (or limit) one another*. This means, however, that the exponent of the direct ratio is unique to the relation between the powers and is not shared by any other direct ratio (such as that between s and t in the average orbital velocity). The Keplerian measure does not by any means lack an immediate exponent – far from it – but that exponent is co-determined by the ratio of powers that constitutes the measure and in that sense is its *own* immediate exponent. This confers an independence on the Keplerian measure which no other realized measure enjoys: it is a relation between quanta, whose powers are determined solely by the "extensive" and "intensive" qualities concerned, and whose particular exponent is then in turn co-determined by those powers. As such, the measure enjoys "a complete *independence*" and is a "*something*" (*Etwas*) all of its own (SL 301 / LS 388).

Since this measure is independent it exhibits *being-for-self*: it is not inherently related to anything other than or external to it, but it is just *itself*. Yet in being

itself, it is not just a measure, or a relation between specified quanta, but it also has an immediate quantum, or exponent, of its own that gives it an immediate determinacy. In the sphere of being, immediate affirmative determinacy is called "reality"; accordingly, Hegel calls the completely independent realized measure a "*real being-for-self*" (*reales Fürsichsein*).

At this point, therefore, measure recovers the being-for-self that it enjoyed when it first emerged. A quantum, prior to the emergence of measure, is "for itself" when it relates to itself in becoming another quantum, that is, when it raises itself to a power of itself. A quantum is initially a measure, however, when it is wholly *self*-relating and does not become another. It is a measure, therefore, when it constitutes quality *by itself*. Accordingly, a measure, initially, stands on its own as a simple, self-relating measure (see SL 288 / LS 371). Such a measure is also determinate because it is a definite quantum or range of quanta; so water remains liquid only between certain definite temperatures.

As the logic of measure unfolds, however, measure ceases to stand alone and comes to relate to something other than it. First, this is simply another external quantum, but then this other proves to be another measure itself and the measure turns into the realized measure we have just been examining. What has now happened is that the realized measure itself has turned out to be wholly independent and "for itself". Furthermore, it also has a determinacy that consists in an immediate quantum. The realized measure thus returns to being a single, stand-alone measure, as it was initially.

Yet it is also very different from the first measure: for it enjoys its independence only because it comprises two measures that are independent of one another and yet *in* their independence form *one* measure that is their "negative unity". These two measures are distinguished from one another by what Hegel calls their "determinacy of measure" (*Maßbestimmtheit*); that is, they are distinguished by the fact that the determinacy of one consists in a specific relation between powers of quanta, whereas that of the other consists in the immediate exponent of a direct ratio. As Hegel puts it, therefore, the realized measure is now an "independent whole" that is "at the same time a repelling within itself into *distinct independent somethings* [*Selbständige*]" (SL 301 / LS 388).

At the start of its logical journey, measure stands alone as a measure all by itself; then it develops into a relation between a measure and a quantum, and into one between a measure and measure. Now once again it stands alone as a measure all by itself, but in so doing it comprises *two* measures that are themselves *relations* between qualities and their quanta. The realized measure, in other words, is now a wholly independent relation between different relations. In this way, however, the realized measure exhibits a form that is no longer that of the realized measure as such. When this form is rendered explicit and considered in its own right, it thus constitutes a new measure that Hegel

calls a "*real measure*" (SL 302 / LS 388). In a realized measure as such, both sides are quanta "of a something and of its quality" (SL 294 / LS 379). Yet those sides are principally the *quanta* of qualities, such as s^3 and t^2, rather than whole things. Furthermore, these sides do not stand alone, but are mere moments of their relation. In the real measure, by contrast, the sides are no longer just moments of a relation, but are themselves separate *relations* between quanta of qualities. As such, they are independent measures or "*distinct independent somethings*" (even though they belong together in one measure). What corresponds to such measures in nature, Hegel maintains, are thus no longer the "abstract qualities" of space and time, but independent physical, "*material*" things (SL 302 / LS 388). Yet these things are not mere things either, for, as we have seen, they are equally relations. The task now, Hegel states, is to consider in more detail the "real" measure that consists in "the relation of such relations" (SL 302 / LS 389).

CHAPTER TWELVE

Real Measure

Measure is the unity of quantity and quality; more precisely, it is quantity that constitutes, or is specific to, quality in some way. Initially (in 1.3.1.A), this measure takes the form of a simple, immediate quantum: it is some particular amount, or range of amounts, that enables something to retain its distinctive quality (such as the temperatures that allow water to remain a fluid). Then (in 1.3.1.B.c) measure mutates logically into a *relation* involving "specified" quanta, or powers, of certain qualities. This relation forms a single measure because it confers a distinctive quality on, for example, a form of motion: so the motion of falling bodies (as opposed to orbiting bodies) is distinguished by the fact that the distance travelled is proportional to the square of the time elapsed.[1]

The two measures, $s = at^2$ and $s^3 = at^2$, are what Hegel calls "*realized*" measures (SL 295 / LS 380). Each, however, contains or coincides with a *direct* ratio between immediately given quanta, the exponent of which is *a*. In the case of $s = at^2$, the direct ratio, governed by *a*, is between s and the first unit of t, as well as between the numerical values of s and t^2; in the case of $s^3 = at^2$, that direct ratio is solely between the numerical values of s^3 and t^2.[2] In each case, however, the direct ratio concerned is the *immediate* measure from which the realized measure is inseparable. The measure governing the planets in our solar system is thus not simply $s^3 = at^2$ in general, but $s^3 = 25t^2$ in particular.[3] A realized measure, therefore, is in fact the unity of *two* measures – one an immediate, direct relation between quanta, and one a relation between (or involving) powers.

Since the exponent of the direct ratio is an immediately given amount, that ratio is independent of the relation between powers and is not governed by the latter. Yet the two relations are inseparable and form *one* measure together. Indeed, in the case of $s^3 = at^2$, the unity of the two relations is especially

intimate, since the value of *a* is in fact partly a function of the two powers themselves: s and t have independently given, and variable, numerical values, but only the quotient of their powers is the constant exponent, *a*. The realized measure thus proves to be both the relation between and the unity of two ratios, and as such exhibits a form that is implicitly that of a *new* measure. This new measure, when it is rendered explicit, is the *real* measure – a single "independent whole" that is at the same time a "relation [*Beziehung*] of measures" that are themselves independent relations (SL 301-2 / LS 388).

Since the two sides of the real measure are no longer just *moments* of a relation – like s^3 and t^2 – but independent relations in their own right, they are, in Hegel's words, "*distinct independent somethings* [*Selbständige*]" (SL 302 / LS 388). Furthermore, Hegel notes, in the initial, immediate form of the real measure the independence of the two sides is itself simple and immediate: the two sides are just separate measures with no further specification. This means, however, that the difference between them is no longer what it is in the realized measure. There one of the measures is a relation involving powers, whereas the other is a direct ratio between quanta.[4] Now, by contrast, the two are separate measures that are just immediately different, and so, as Hegel puts it, they are initially no more than "immediate measures" (SL 302 / LS 389). As such, they must both be ratios with immediate exponents, or *direct* ratios.[5]

The realized measure develops to the point at which it exhibits the form of a new measure. When that new measure is rendered explicit and considered in its initial immediacy, however, it proves to be a relation between two direct ratios, rather than between one direct ratio and a ratio of powers (as in the realized measure). In Hegel's words, "since the sides which now constitute the measure-relation [*Maßverhältnis*] are themselves measures, but at the same time real [*reelle*] somethings, their measures are, in the first instance, immediate measures, and the relations in them direct relations" (SL 302 / LS 388-9).

REAL MEASURE AS THE RELATION BETWEEN TWO RATIOS: SPECIFIC GRAVITY

Hegel begins his account of the real measure (in 1.3.2.A.a) by considering the two immediate measures that it contains. Each of the latter, as we know, is a something that is also a relation. More specifically, we are told, each is a "measure-relation between quanta, to which qualities are also attached [*zukommen*]" and so is also a relation between these qualities (SL 303 / LS 390). Each immediate measure must relate qualities, as well as quanta, to one another, since each is derived logically from the realized measure, which is the "quantitative relating of *two* qualities to each other" (SL 295 / LS 379). The particular qualities that are now in relation, however, are rooted in the logical

structure of the new *immediate* measure. They reflect the fact that such a measure is (a) an independent something (and so qualitative) and (b) a ratio (and so quantitative). Indeed, the two qualities related *in* the immediate measure are just these two different aspects, or qualities, *of* the measure itself.

Consider first that an immediate measure that is one side of a real measure is an independent *something* in its own right. Earlier in the *Logic* we learned that what distinguishes "something" (*Etwas*) from mere "determinate being" (*Dasein*) is self-relation or "being-within-self" (*Insichsein*): something is not merely determinate – this, *not* that – but *itself* (SL 89 / LS 110).[6] One aspect, or quality, of the immediate measure is thus "the something's *being-within-self* [*Insichsein*], according to which it is something for itself [*Fürsichseiendes*]" (SL 303-4 / LS 390).[7] The immediate measure, however, is also quantitative, since it is a relation between quanta; and, as we have seen in earlier chapters, the distinctive quality of quantity is *externality* (*Äußerlichkeit*) (see SL 271 / LS 350). The latter is thus the other quality of the immediate measure in a real measure (SL 304 / LS 390). The immediate measure, therefore, is the relation between the two qualities of being something for itself and being quantitative or "external".

Yet being something and being quantitative or external are not simply distinct aspects of the immediate measure, but they also coincide with one another. This is because the measure is not just a qualitative, but also a *quantitative*, something, or a quantum. (The measure is a *relation* between quanta; as a direct ratio, however, it has an immediate exponent, and this exponent is the particular quantum that constitutes the measure.) The measure's quality of being something – in contrast to its being external – must, therefore, itself coincide with its being a quantum. Now a quantum is a determinate unit of quantity and, more precisely, a number with a definite amount (see SL 168-9 / LS 213-14). The measure's quality of being something must, therefore, coincide with its being a definite amount or "*plurality*" (*Menge*) of units: it must be *something* in and through being an *amount*. Such quality is found, for example, in a thing with *mass* – that is, with "material parts" (*materielle Teile*) – or, when mass exerts pressure due to gravity, in a thing with "weight" (*Gewicht*): for it is precisely through its mass, which is itself a definite amount of matter, that such a thing is an independent something (SL 304 / LS 390).[8] Accordingly, whereas a realized measure is exemplified by the relation between space and time, the immediate measures in a *real* measure are exemplified by things that have mass or weight.

Note that the other quality of the immediate measure – externality – does not itself confer "being-within-self" on the measure and so does not make the latter an independent something. It must, therefore, be a mere *moment* of such a something. Hegel sees this moment of externality exemplified in the feature of mass-bearing, material things that makes them external to themselves and so extended, namely *space* (SL 304 / LS 390). In Hegel's view, therefore, the

immediate measure we are considering here is best exemplified by the relation between the mass of a thing and the space it takes up, or its volume. This measure, however, is a relation not just between qualities, but also between definite quanta of those qualities. The measure is exemplified, therefore, by the direct ratio between the *particular* mass, or weight, and the *particular* volume of the thing. This ratio is identified by Hegel with the "specific gravity" (*spezifische Schwere*) of the thing, though it constitutes, more precisely, the thing's density or specific weight (in German, *spezifisches Gewicht*).[9] In Hegel's words, "these qualities are quantitatively determined, and their relation to one another constitutes the qualitative nature of the material something – the ratio of weight to volume" (SL 304 / LS 390). (In what follows one should not forget that density is merely an *example* of the ratio that forms one side of a real measure. For ease of discussion, however, I will sometimes identify that logical ratio with density itself.)

Now in a direct ratio, as we recall (from 1.2.3.A), one of the quanta must count as the amount and one as the unit (even though the unit itself will contain an amount). That is what allows us to consider the ratio between, say, 8 and 2 to be that of 4 : 1. So in the direct ratio of mass or weight to volume, one of the two must be the unit to the other's amount. Since the mass consists, by definition, in an *amount* of parts, Hegel contends that the volume must provide the unit. Density, therefore, is measured in mass per unit of volume (though, in his philosophy of nature, Hegel maintains that these terms can also reverse their functions and density can be measured in terms of volume per unit of mass).[10] Note, by the way, that such density, for Hegel, is not just an external measure or "standard" applied to things by us; it is a measure that belongs to things themselves.

The *immediate* measure that forms one side of a *real* measure thus consists in the direct ratio between the amount that gives something its independence and a unit of its externality; in more concrete terms, the immediate measure consists in the ratio between a thing's mass and a unit of its volume, or in its density. Each of the sides of this ratio is an immediate quantum on its own, but one of them – namely, the amount – also provides the *exponent* that defines the ratio itself: so a ratio of 4 : 1 between mass and volume yields the defining quantum, 4, for the ratio. The immediate measure is thus a thing (or "something") that is also a ratio, but what defines it, and gives it its independence, *as a ratio* is specifically the *amount* it contains. Accordingly, Hegel calls this amount, or exponent, the "specific quantum of the something" (SL 304 / LS 391).

Now the first measure examined by Hegel in the *Logic* (in 1.3.1.A) is also called a "specific quantum", but the exponent of a ratio such as density is significantly different from this. The first specific quantum constitutes, and sustains, some distinct quality: it consists, for example, in the temperature, or

range of temperatures, that keeps water liquid. The realized measure is a ratio, rather than a mere quantum, but its two sides are still quanta of distinct qualities (such as space and time). In the real measure, which is also a ratio, the two sides are no longer just quanta – whether simple or raised to a power – but are themselves *ratios*, such as densities. The sides of these latter ratios, however, are – like the sides of a realized measure – quanta of distinct qualities: so density is the ratio between a quantum of *mass* and a quantum of *volume*. Yet note that density *as such* – *as a ratio* – is not a quantum of a distinct quality in the same way. As we shall now see, it is, rather, *itself* the defining quality of the thing to which it belongs in the real measure, and the same is true of its exponent.

Density is a ratio that forms one side of a real measure, and the two sides of such density itself are distinguished by their quantity and by their quality (which is either mass *or* volume). The two sides of the *real measure*, however, have the same quality (since each, equally, has both mass *and* volume), so they are distinguished qualitatively by nothing but the *ratio*, or density, in which each consists. This quantitative ratio alone, therefore, gives each side of the real measure its distinctive quality and thus its real independence as a thing. In Hegel's words, this ratio or "relation" (*Verhältnis*) "constitutes the qualitative nature of the material something" (SL 304 / LS 390).[11]

This means in turn that this thing's distinctive quality must also consist in the amount or quantum that is the exponent of its ratio. Yet here we face a problem: for how can a quantum simply *be* the quality of something? A quantum can certainly sustain a quality that is distinct from it, and so be a measure, but how can it *be* a defining quality itself? It can do so, Hegel suggests, by taking on, as a mere quantum, the logical form of a quality. Yet since it is by definition a simple, immediate quantum, and the exponent of a direct ratio, it cannot become "qualitative" by raising itself to a power of itself. Hegel argues, however, that a quantum can become qualitative in another way.

In the sphere of quality, things and their defining qualities, unlike numbers, are not simply determinate in themselves, but they also owe their determinacy to the *limit* that marks off each one from – while binding it to – something else.[12] The exponent of a thing's internal ratio thus becomes the distinctive quality of that thing when it, too, is compared and contrasted explicitly with the exponent of another thing: for, in this way, it ceases being a simple number and becomes the exponent, or specific quantum, of *this* ratio, rather than *that*. Hegel puts the point like this: "this exponent is the specific quantum of the something, but it is an immediate quantum and this is determinate [*bestimmt*] – and with it the specific nature of such something – only in *comparison* [*Vergleichung*] with other exponents of such ratios" (SL 304 / LS 391). It is through this comparison, therefore, that the exponent "constitutes the *specific intrinsic* determinacy [das spezifische An-sich-*Bestimmtsein*], the inner characteristic measure of something".[13]

To repeat: the exponent of the ratio in a thing is by itself simply an amount or number. It gives the thing a specific, distinguishing quality, however, when it is compared – explicitly – to the exponent of the ratio in another thing (which may be its current counterpart or something else). This quality is different from the two qualities *in* the ratio itself: for it consists simply in the exponent *of* that ratio, compared with that of another. So, using Hegel's example: the thing has its own mass and volume; it has also a given density, which is the ratio between its mass and volume, and that density has an exponent; this exponent then confers a distinctive *quality* on the thing, when it is compared explicitly with another density's exponent (that is, with a different *amount* in a common unit of volume).

Since both exponents are fixed quanta, they can stand in a direct ratio to one another in which one serves as unit and the other as amount. This ratio between the two exponents (and between the ratios, or immediate measures, to which they belong) is now the *real* measure that is present here – the most developed form of measure to have emerged so far in the *Logic*. This real measure, as a direct ratio between quanta (which are themselves exponents of ratios), of course has its own exponent that depends on which of the two quanta is taken to be the amount and which the unit in the ratio.[14]

An immediate measure such as density thus necessarily has a double status: on the one hand, it is the measure of the specific thing to which it belongs; on the other hand, its exponent can be a measure in the full sense, and so confer a distinct quality on the thing, only in explicit relation to the exponent of another thing's density. When the density of a thing is set in direct ratio to that of another thing in this way, such that the latter counts as the unit in the ratio, the density of the first thing becomes its *specific gravity*. The specific gravity of something is thus its density in explicit *relation* to that of a reference substance. For solids, the latter is usually water, the density of which is held to be 1; so pure gold has a specific gravity of 19.32, since it is 19.32 times as dense as water, and lead has a specific gravity of 11.34.

Speculative logic cannot predict these particular numbers, since they are – from the philosophical perspective – immediately given and so contingent. It shows, however, that things must contain ratios such as density, and that these ratios are the measures of things just as much as the "specific quanta" that sustain the qualities of those things. (So the measure of a solid metal lies in its density, as well as in the range of temperatures below its melting point.) Logic also shows that the exponents of such measure-ratios must stand in direct ratio to one another, and that density is thus a proper measure of something only when it is conceived – in explicit relation to the density of, for example, water – as the thing's specific gravity. Indeed, logic shows that such specific gravity is the *real measure* through which things enjoy, and manifest, their concrete specificity.[15]

Yet logic also takes us further. A measure, such as density, "rests on" – and its exponent is – a simple quantum, namely the amount of the mass per unit of volume. This quantum, as "an external, indifferent determinacy", is, however, necessarily *changeable* (SL 304 / LS 391). This leads us to a further form of real measure.

REAL MEASURE AS THE COMBINATION OF TWO MEASURES

The idea that the quantum constituting the measure of something is changeable is not itself new. We encountered such variability in the first form of measure: the simple "specific quantum" of something (in 1.3.1.A). Take, for example, the temperature of water: any temperature from 1° C to 99° C can be regarded as constituting or belonging to the measure or specific quantum of water, since it keeps water liquid. That temperature, as a simple quantum, can, however, increase or decrease. Furthermore, it can increase or decrease to the point at which water ceases to be liquid. Water can, therefore, be transformed by the change in its quantum into something – steam or ice – with a new specific quantum or range of such quanta.

The measure we are now considering – an immediate measure (such as a density) in direct ratio to another such measure – also has a quantum that is necessarily changeable. What it means for it to be changeable is, however, somewhat more complicated than in the case of the first form of measure. This is because the quantum of the current immediate measure is the *fixed* exponent of the ratio, or density, in which the measure consists.

A purely quantitative ratio can, of course, change into another ratio and so acquire a new exponent. Any particular (direct or inverse) ratio, however, has a fixed exponent that makes it *this* ratio, rather than another – 4 : 1 as opposed to 5 : 1 (see SL 272-7 / LS 351-8). The "immediate" ratio that forms one side of a real measure is just such a particular ratio: the direct ratio that constitutes the distinctive quality of a something (when compared explicitly to the ratio of another thing). It is the unchanging ratio that makes something *this* thing and not another, and so it must have a fixed exponent.

Now, as we know, every something as such – as a *something* – is subject to change and can even be taken beyond its limit and become another something altogether (see SL 92, 101 / LS 114, 125-6). As one side of a *real measure*, however, something is a distinct, independent something in relation to another (for example, a piece of lead in relation to a unit of water), and in this respect has a specific identity consisting in a specific ratio with a *fixed* exponent. Accordingly, if the amount, or mass, of the thing changes by becoming bigger or smaller (which, as an amount, it is able to do), the volume of the thing must also change so that the ratio between them is preserved: the more mass there is,

the more space it occupies. The exponent of the thing's *density*, namely the mass per unit of volume, thus remains fixed, despite the change in mass.[16]

Yet this very exponent, as a *quantum*, is necessarily changeable as well – "despite its inner determination as measure" (SL 304 / LS 391) – since it lies in the nature of a quantum to be changeable.[17] Since, however, this exponent, as one side of a real measure, is itself fixed, it cannot change of its own accord; it is and remains the exponent it is. So what can make such an exponent change? What can make it become a different exponent, the exponent of a different measure? This can only be the exponent or quantum of *another* measure (another density) to which it stands in relation – "a quantum which is at the same time equally the exponent of such a specific ratio" (SL 304 / LS 391) – for, at this stage in the *Logic*, such measures are what there is.[18] In the doctrine of quality, we saw that every something has a constitution that can be altered by the other to which it relates (see SL 96-7 / LS 120-1). Now we see that immediate measures – the sides of a real measure – must also be changed by other such measures. In their case, however, they must be changed by another, not simply because they have a constitution, but because in themselves they are fixed and yet, as quanta, *they must change*. They must become other than they are, but they can do so only through the agency of that which is itself other than they are.[19]

The density (or specific gravity) of a thing is fixed for that thing, but it is necessarily open to being changed by the density (and exponent or quantum) of something else. As Hegel puts it, change has to come to a measure, such as density, "through the externality of the quantum", that is, through a quantum *outside* the measure itself (SL 304 / LS 391). Hegel notes that such a density is not changed merely by taking on a further "amount [*Menge*] of material", but "holds out" (*hält aus*) in face of the latter; as we have just seen, if we simply increase the mass and volume of a thing, the thing's density, which is the fixed ratio between the two, remains unaltered. Yet this density is vulnerable to being altered by a *different density*, and necessarily so. Such change, however, is not brought about by the mere fact that a density stands in a direct ratio to another density; it occurs only when the other density actually modifies the first. Since each density has an exponent that is just a simple quantum, the second can modify the first only by *adding* itself to it. The logical requirement that measures, such as densities, be changeable thus itself requires these measures to *combine* with one another to form a new measure. In Hegel's words, logic requires that there be "two things of different inner measure that stand in relation and enter into combination [*in Verbindung treten*] – such as two metals of different specific gravity" (SL 304 / LS 391).[20]

Note the subtle shift that has now occurred in the structure of the real measure. A real measure is the relation *between* two immediate measures, but it has now also proven to be the *combination* of such measures. In this way the unity that is inherent in the idea of a real measure has become fully explicit.

The real measure was made necessary, logically, because the realized measure took the form of a new measure: it ceased merely being the ratio between two different *moments* (such as s^3 and t^2) and proved to be the ratio between two *independent* measures (each of which is itself a ratio). These two measures, however, also formed a unity which itself enjoyed "complete *independence*" (SL 301 / LS 388). When this logical structure was then rendered explicit and considered in its own right, the *real*, rather than merely *realized*, measure emerged. The real measure is thus a ratio between two independent measures that is also the unity of those measures.

In his account of the real measure, as we have seen, Hegel starts by examining the two independent, and immediate, measures that belong to it, and shows that each in itself is a direct ratio between two qualities of a thing (such as mass and volume). He then argues that the exponent of each ratio (and thereby each ratio itself) must be explicitly *related* to another such exponent – its counterpart or something else – and that a thing's density must therefore be its specific gravity. We have now seen, however, that any such ratio must actually be combined into a unity with another one: for only in this way can its *fixed* exponent undergo the *change* that its status as quantum requires it to undergo. At this point, therefore, the unity inherent in the idea of real measure as an *independent* "relation of measures" becomes fully explicit (SL 302 / LS 388). This, one should note, means that a real measure is itself inherently changeable. Such a measure has a fixed exponent that governs the ratio between its constituent measures; but it is the true unity of those constituent measures only when they combine to produce a *new* measure with a *new* exponent.

Hegel acknowledges that in nature certain conditions have to be met for such a combination to occur: a metal, for example, cannot simply combine its density with that of water, and two metals, such as copper and zinc, have to be heated and melted to be combined (into brass).[21] In the *Logic*, however, we are not concerned with such natural conditions; logic shows simply that an immediate measure that is itself a ratio, such as density, must (be able to) combine with another immediate measure. This necessity is grounded in the fact that each such measure is indeed an *immediate* measure. Each measure is a ratio with a fixed exponent, but the latter is also an immediate and so *changeable* quantum; yet precisely because it is fixed in itself, it must be changed by the exponent of another measure that adds itself to it.

Logic also has more to tell us about what must go on in such change. First, since a measure has a fixed exponent, it must "preserve itself" in the very process of being changed by another measure. Indeed, Hegel points out, "each of the two measures preserves itself in the alteration" (SL 304 / LS 391). Second, in preserving *itself* each necessarily negates in some way the quantum that is added to it by the *other* measure, and thereby qualifies or "specifies" the latter. This echoes the relation we saw above between a "specifying measure",

such as a body's specific heat, and a quantum that is added to the body from the outside.

We recall that an external measure can take the form of a simple standard (*Maßstab*) in terms of which something else is to be measured; but such a measure is in fact merely a *quantum* that provides the unit for another quantum (see 1.3.1.B.a). A measure is a proper *measure* when it sets not just its quantum, but also its qualitative nature, in relation to another quantum; and it does that when it imposes a qualitative limit on, and so negates, that other quantum. When a quantum is added to such a "specifying" measure, therefore, what the latter takes up is not just the quantum that is added but one that has been qualified and altered by the measure itself (see 1.3.1.B.b). This same act of qualification also takes place here in the real measure: for, as two measures are combined, each adds its own quantum to that of the other, but each preserves itself at the same time by asserting itself against, and so qualifying and limiting in a way that is specific to it, the quantum that is added to it. The combination of the two measures thus involves "a reciprocal specification" (SL 304 / LS 391).[22]

The process of change just outlined would appear, however, to be contradictory. A measure is changed by combining with another; indeed, both are changed by their combination. Since, however, each measure preserves itself in the process, their combination must entail the simple addition of the *fixed* quantum of each to that of the other. Yet in preserving *itself* in face of the other, each also specifies, limits and so *alters* the quantum added to it by the other (and so in this respect, paradoxically, neither quantum is in fact preserved in their combination). Note that both things must be true: the measures must both be preserved and not be preserved in their combination. Yet how can both happen at once? This is possible, Hegel maintains, because the combination of the two measures actually involves the combination of *two* different sets of quanta. Density (for example) is the ratio of mass (or weight) to a unit of volume, so when densities are combined both the masses of the things *and* their volumes are combined.

If the quanta of the two measures were only to preserve their fixed quanta in their combination, the exponent of the new density would be easy to calculate, since one would simply add the two mass-quanta together and divide the result by the sum of the two volume-quanta (see SL 304-5 / LS 391). So if we were to add one thing with a density of 4 per unit of volume to another with a density of 6 per the same unit of volume, we would get a new thing with a density of 10 per 2 units of volume, or 5 per unit. The exponent of each density would thereby *change* into the exponent of the new combined density; yet we would not have taken account of the moment of reciprocal specification in the combination of the densities. Such combination cannot, therefore, merely be a matter of adding the two masses and then the two volumes together and dividing the former sum by the latter.

Yet, Hegel maintains, the values of the two *masses* must be simply *added* together, because they must preserve themselves as *fixed* quanta. This is due to the fact that, in the real measure, the mass of each thing (per unit of volume) – whose quantum is the exponent of its constitutive ratio – is what gives it its fixed, *independent* identity and so makes it the particular thing it is (see 2: 279). As Hegel puts it, the mass (or weight) is the aspect of each thing "which, as being-for-self, has acquired a fixed determinate being and with that a permanent immediate quantum" (SL 305 / LS 391-2). Since the thing owes its independence and distinctive quality as a thing to its mass (per unit of volume), that mass cannot be changed when it is combined with the mass of another, but must be preserved; both masses, therefore, must be simply added together. This is not true, however, of the volumes of the two things. Each thing has its own volume (and volume per unit of mass), but volume itself is not what makes it a distinct, independent *something*; it is, rather, a *moment* of the thing that gives it the quality of being external to itself or extended. The volumes of the two things do not, therefore, have to remain fixed and be preserved, when the two things are combined. It is thus these volumes that are specified and altered in such combination. Note by the way, that Hegel considers this alteration of volume to be logically necessary *and* to be empirically observable. "To sensuous perception", he writes, "it may be striking that the mixing of two specifically different materials should be followed by an alteration – usually a diminution – in the total volume"; but such alteration is made necessary logically by the nature of measure itself (SL 305 / LS 392).

To repeat: when two densities are added together, it is not simply a matter of adding the two masses and the two volumes and then dividing the first sum by the second, for logically there should be an alteration of the overall volume. This is not to say that this will always be observable in nature; but in some cases it will be. One obvious example of such alteration is provided by dissolving salt crystals in water: for, in this case, the mass of the resulting solution is the sum of the two masses, but the resulting volume (when the saturation point is reached) is less than the simple sum of the two volumes.[23] Note, however, that, *pace* Ulrich Ruschig, Hegel is not just led by such *examples* to claim that the volume is altered when densities are combined. He is led to this thought by the *logic* of measure itself – a logic that thus provides a rational (though not natural-scientific) explanation for the observed phenomena.[24]

REAL MEASURE AS A SERIES OF MEASURES

A real measure is a direct ratio between two independent measures that are themselves direct ratios. So specific gravity, for example, is the ratio of one density, which is itself a ratio between mass and volume, to another such density which has 1 as its exponent (such as the density of water). Yet, logically, a real

measure must be more than just a relation between, and become a unity or combination of, such measures (though in nature specific conditions may be required for such combination to occur) (see SL 304 / LS 391). As we have seen, a measure, such as a density, must combine with another, because its exponent is fixed in itself and yet, as a quantum, must be subject to change, and so must be changed by *another* measure. Both measures, indeed, change through being combined, since the two together form a new measure with a new exponent.

In the doctrine of quality, we saw that every something is changeable within itself (though it is also changed by the other to which it relates) (SL 92, 96-7 / LS 114, 120-1). In the doctrine of quantity, the concrete quantum is shown to be changeable, because it is essentially self-external and so has its own determinacy *outside* itself in another quantum: things are thus necessarily susceptible to becoming bigger or smaller (SL 189 / LS 240). We have now seen that things as measures, understood as direct ratios with fixed exponents, are also subject to change: as Hegel puts it, each such measure shows itself "not to be something fixed in itself, but, like the quantum as such, to have its determinacy in other measure-relations" (SL 305 / LS 392). These measures change, however, not by becoming other measures (that is, other ratios) themselves, but by *combining* with other measures.

At the start of 1.3.2.A.b, Hegel then notes that if two things were determined solely by their "simple quality", they would just "sublate themselves" (*sich aufheben*), or cancel one another out, when combined (just as mixing red paint with blue produces a colour that is neither, namely purple) (SL 305 / LS 392). Something as a measure-relation, however, preserves its independence when combined with another such measure and is thus not cancelled by the combination. More specifically, the quantum of each measure continues in the new measure, insofar as the latter results from simply adding the two quanta together. What emerges is, indeed, something new; but insofar as the new measure is the sum *of* the two original measures, the latter can be said to "preserve" themselves in it. Yet each measure also functions as a "specifying moment" in the new measure; that is to say, each *alters* the other to a greater or lesser degree (SL 305 / LS 393). This is why, when two densities are combined, the two masses are added together but the new volume is not the simple sum of the two volumes. The new density is thus not *just* the straightforward unity of the two original measures in which the latter can be said to preserve themselves. Indeed, each measure in a new combined measure actually *loses* the fixed, independent character that resides in its particular ratio, since the new measure makes that character invisible. The new measure does not manifest the fact that it combines *this* particular ratio with *that* one, but it is simply a new and different ratio that could have arisen in a variety of ways. Accordingly, Hegel writes, "the specific peculiarity [*Eigentümlichkeit*] of something is not expressed

in only one measure-relation formed by it and another something" (SL 306 / LS 393).

Yet each measure is an *independent* measure in its own right and must preserve that independence in the process of being changed. Since it does not do so when it enters into just one combination with another measure, it must do so by also entering into further combinations with other things. So far, as we have seen, the logical structure of measure has restricted the latter to a relation between just two elements (which take different forms in different measures), a relation grounded ultimately in the distinction, inherent in measure itself, between quantity and quality. Logically, however, a measure must relate to more than one other thing, since it is not just *something* in relation to *another* but also a quantum that is a *one*, and, as we know, a one is always one of many, indeed one of indefinitely many. This aspect of the measure now becomes explicit for two reasons. First, as just noted, a measure, such as a density, must combine with more than just one other if it is to preserve its independence in the process of combination. Second, each measure, as a *quantum*, is essentially indifferent to the one with which it combines, and so can always combine with another, or another (provided, of course, that the conditions are right). This means in turn that the new exponent produced when two measures combine is itself just one "indifferent" quantum among *many* possible quanta (and, Hegel remarks, the exponent of the new measure shows itself to be "indifferent" precisely *when* each of its component measures also forms combinations or "neutralizations" with others) (SL 306 / LS 393). Note that a measure, due to the moment of simple *difference* at its heart, still relates to, and combines with, one other measure at a time, and so, unlike the pure one, is not required to unite with several others at once (though it may, as a matter of fact, be able to do so).[25] It unites with several other measures, therefore, by entering into a *series* of combinations with different single measures.

The combination of one measure with a series of other measures, Hegel writes, "yields different ratios that therefore have different exponents" (SL 306 / LS 393). So adding a density of 2 : 1 to other densities, such as 3 : 1, 4 : 1 and 5 : 1, produces a series of new densities with distinct exponents (which, however, are not necessarily just 2.5, 3 and 3.5, due to the change in volume). This series of exponents is, on the one hand, just a series of different amounts; but, on the other hand, it is the distinctive series that is produced by *one* particular measure when it combines with others. It is thus "*a series of specific ways of relating to others*" that is characteristic of that one measure. As such, it expresses the independent character and quality of that measure in a way that a single combination does not.[26]

A measure-relation, such as a density, taken by itself has its own fixed exponent which confers a distinctive quality on the thing concerned when it is conceived in explicit relation to that of another density (such as the density of

water or air). This exponent, as a quantum, is also necessarily changeable. Since it is fixed in itself, however, it can be changed only by combining with that of another, different density. In the process, as we have seen, each density is "specified" and altered by the other density with which it combines. Indeed, it *loses* its distinctive, independent character altogether in the unity it forms with another. It preserves and expresses its independence, however, in the distinctive *series* of ratios and exponents that it forms with other densities: for, even though it is "specified" and altered in different ways in each combination, that series itself is distinctively *its own*. The exponents in the series are all quite different amounts or quanta, but they all have *one* density in common, since they all result from combining *that* density with others. This density thus stands as their common "unit" (*Einheit*), and the series as a whole serves to distinguish the density from another that produces a different series.

> The independent measure truly distinguishes itself through the *characteristic series* of exponents which it, taken as unit, forms with other such independent measures; for another of those measures, when also brought into relation with the same ones and taken as unit, forms another series.
>
> —SL 306 / LS 393

Yet if the series of exponents, produced by a measure when it combines with other measures, is to preserve and express within itself the distinctive, independent *quality* of that measure – and to do so fully and explicitly – it must consist in more than mere *quanta*. That "more" in turn can only lie in the *order* in which the quanta are arranged – the fact that the measure concerned, when combined with other measures A, B and C, produces the series 3, 4 and 5, rather than 5, 3, 4. It is thus not just a series of quanta as such that distinguishes the quality of a measure from that of another measure (that produces different quanta), but the distinctive relation between the quanta in the series. As Hegel puts it, "the relation [*Verhältnis*] of such a series within itself now constitutes the qualitative aspect of the independent measure" (SL 306 / LS 393).[27] This does not undo what we learned earlier in 1.3.2.A.a: it is still the case that the original exponent of a measure confers a distinctive quality on the latter when it is compared explicitly with that of another measure (see SL 304 / LS 391). Now, however, a measure has multiple exponents that are produced when it combines with many other measures, and these different exponents stand in a certain relation to one another. This relation between, or order of, a measure's multiple exponents thus now also expresses the distinctive quality of that measure.

The difference between two independent measures is thus not simply one between two series of variable quanta produced when each measure combines with other measures; it is a difference between a series of quanta that stand in

a definite, fixed relation to one another and another such series. Since this difference is between *fixed* series of quanta, it takes the form of a complex direct ratio. A simple direct ratio is between fixed *quanta*, such as 4 : 1. The initial form of real measure is then a direct ratio between measures that are themselves direct *ratios* (e.g. between mass and volume). Now we have a further form of real measure that consists in the direct ratio between measures that are themselves expressed in definite *series* of direct ratios and their exponents.

In the first real measure, exemplified by specific gravity, two ratios – in this case, densities – stand in relation to one another as, say, 4 : 1 and 1 : 1, where the first number in each case represents the mass and the second number the volume. The real measure itself can thus be expressed as a simple ratio 4 : 1, because its two component ratios share a common unit, namely a volume of 1: the density of X is four times that of the same volume of Y. If two *series* in the new, and more complex, real measure are also to stand in a direct ratio to one another, they, too, must have a common unit. That common unit, however, is not to be found in the original exponents of the two measures, X and Y, that produce the two series, for each exponent provides the unit only for the particular series of exponents to which *it* gives rise. Let measure X, with an exponent of 3, produce *this* series of exponents with other measures, and let measure Y, with an exponent of 5, produce *that* series. These two series are different, but as such they are not directly comparable, and cannot be set in direct relation to one another, because they are not different series of the *same* unit (in the way specific gravity relates two densities with the same unit of volume). They are simply different, incommensurable series produced by different units – units with different amounts – and so are what Hegel calls "*indeterminate others*" (*unbestimmt andere*) (SL 306 / LS 394).[28]

Yet each of the two measures not only has its own original exponent or specific quantum, but its specific nature or quality as a measure is displayed in the distinctive order of the series to which it gives rise or, in Hegel's words, in "the relation which the exponents of the ratios [*Verhältnisexponenten*] in the series have to one another" (SL 306 / LS 394). It is such order or relation, Hegel contends, that can provide the "common" (*gemeinschaftlich*) unit that enables the two series, and thereby the two measures, to be set in direct relation to one another. Yet how can this be, since each series is characterized by its *own* distinct order? It is possible, if there is an aspect of such an order that is actually *the same for both measures*. This common element cannot consist in the specific series of numbers that characterizes either measure, since that will be (or is likely to be) different in each case. A common element will, however, be present if each series of numbers is produced by combining the measure concerned with the *same* set of other measures *in the same order*.[29]

To see what Hegel has in mind, let our two measures, X and Y, again each form a series of exponents with other measures, but let those other measures in

each case now be A, B and C in that specific order. Each series of exponents will (or may) comprise a different sequence of numbers; yet each will be produced by combining X or Y with the *same* ordered series of other measures. In this way, that series itself – A, B, C – constitutes the common unit that enables the two other measures, X and Y, to be set in direct relation one another. X in relation to A, B and C will produce *this* ordered series of exponents, and Y in relation to the same sequence of other measures will produce *that* ordered series of exponents. The two different series will thus have a common standard of comparison. It is this idea, I think, that Hegel presents in the following remark: the relation between the exponents in a given series contains a *common* "determinate unit" for two measures (X and Y) "only in so far as the members of the series have the same [relation], as a *constant* relation among themselves, *to both* [measures]" – a constant relation that is determined by, indeed that just *is*, the ordered sequence A, B, C (SL 306-7 / LS 394).[30]

X and Y are independent measure-relations (for example, densities) with their own original exponents that take the form of simple quanta. Each measure, however, expresses its distinctive nature and quality by combining with other measures to produce a specific series of new exponents. These new exponents are also simple quanta, but they stand together in an ordered series that is characteristic of the measure concerned. The individual measure – X or Y – provides the common unit that enables the exponents *within* its series to be compared to one another: so one exponent results when X combines with A, another when X combines with B, and so on. As Hegel puts it, each of the two independent measures, X and Y, taken separately "is the unit of the ratios [it forms] with the members standing over against it" – with A, B and C – "which are the amounts relative to that unit and which thus represent [*vorstellen*] the series of exponents" (SL 307 / LS 394).

Yet the series of exponents produced by X and Y also contain within them the unit that enables *X* and *Y* themselves to be compared with one another. They do so, however, not just insofar as they comprise certain *numbers* or *quanta*, but insofar as they comprise combinations of X or Y with the *same ordered series of other measures*: A, B, C. The two different series of exponents can thus be compared as different versions of the same, constant series, one of which owes its character to the exponent or amount of X and one to the amount of Y. In Hegel's words, "this series is the unit for the two measures which when compared with one another are related as quanta"; and as such quanta, the two measures "are themselves different amounts of their just indicated unit" – that is, different amounts, X and Y, that together with the common unit, "A, B, C", produce the two different series of exponents (SL 307 / LS 394).[31]

Hegel now reminds us, however, that A, B and C are themselves independent measures. They unite with X and Y – and indeed with other measures (for a

measure, as *one*, is always one of *many*) – to form the distinctive series that belong to the latter; yet they are also measures in their own right:

> But further, those measures, which, together with the two or rather indefinitely *many* that stand over against them and that are compared with one another [*unter sich*], yield the series of exponents of the relating [*Verhalten*] of those many, are equally independent in themselves, each being a specific something with its own proper measure-relation.
>
> —SL 307 / LS 394[32]

A, B and C thus form their *own* distinctive series of exponents with X, Y and Z. Moreover, since each has its own quantum, each constitutes the "unit" that connects the members of its own distinctive series. Where the unit of a given series is A, therefore, the exponents in that series will be XA, YA and ZA (rather than XB, YB and ZB).[33]

Now, as we saw above, X, Y and Z and A, B and C are themselves all ratios, as well as quanta that are the exponents of those ratios. They are in fact, therefore, X : 1 and A : 1, or $\frac{X}{1}$ and $\frac{A}{1}$. When such ratios are combined, the result is thus not just X + A, but $\frac{X+A}{2}$ – or, more precisely, it is $\frac{X+A}{(2-v)}$, where *v* represents the amount that is to be subtracted from the combined units of the two ratios (e.g. the combined volumes). Nonetheless, each exponent in the series produced by different measures results from *combining* such measures, and so can be expressed, as we have done above, as XA, YA, ZA and so on. When the exponents are represented in this way, it is evident that they have the measure that supplies their unit – in this case, A – in common, and are differentiated by the other measures – in this case, X, Y, Z – with which that unit is combined. It is those other measures, therefore, that determine (with the qualification just made) the different exponents for a given unit. This, I think, is what Hegel has in mind when he writes that the second set of measures – A, B, C, in our example – "are similarly to be taken each as a unit so that they have a series of exponents *in* [*an*] the two or rather the indeterminate plurality of measures, which are named first and compared only among themselves [*unter sich*]" – that is, X, Y, Z (SL 307 / LS 394). It should be borne in mind, though, that the numerical value of the exponents XA and YA is not the same as that of X and Y themselves, but the result of combining the latter with A.

Hegel's account of the series of exponents produced by combining measures is, in my view, clear and coherent, but his text is complex and abstract. The abstractness of Hegel's text is due principally to the absence of examples. When those examples are supplied, however, the logical structure that he is describing becomes much easier to comprehend. The thing to bear in mind when reading the final paragraph of 1.3.2.A.b.2 is that there are two different *series* of

measures – X, Y, Z and A, B, C – to be considered. This is made necessary by the logical development that we have been tracing.

A real measure, we recall, is initially the ratio between *two* measures that are themselves ratios. The logic of measure, however, soon changes that number. First, it requires the measures, not only to stand in relation to one another, but also to combine to form a *third* measure. This reflects the fact that the real measure is inherently the unity of, as well as the relation between, two measures. Second, the logic of measure requires the first two to be, explicitly, two among indefinitely *many* measures. This is, ultimately, because a measure, as a quantum, is not just *something* in relation to something *else*, but a further form of being-for-self or the one, and the one is necessarily one of many. So each of the two original measures – X and Y – has to be thought as combining with several other measures, which we call A, B and C. Furthermore, X and Y are distinguished from one another, not just by their specific quanta or exponents, but more properly by the distinctive ordered series of further measures and exponents they form with A, B and C.

We have now seen that A, B and C in turn combine with X and Y – and, indeed, with many others – to form distinctive series of their own. There are thus not only two series of exponents, produced by X and Y, but two *sets* of series, produced by X, Y, Z and by A, B, C respectively. This is not to deny that further levels of complexity can be added to the picture Hegel is presenting. What is important, however, is that the logic of real measure requires there to be, minimally, first two *measures* in relation that are themselves ratios, then two *series* of measures in relation, and then two *sets or series* of series.

We can represent these two sets or series of series as follows:

Set 1 (in which X, Y, Z are the units):

X → XA – XB – XC

Y → YA – YB – YC

Z → ZA – ZB – ZC

Set 2 (in which A, B, C are the units):

A → XA – YA – ZA

B → XB – YB – ZB

C → XC – YC – ZC

Reading from left to right after each arrow, we have the series of exponents that belongs to a particular measure. The different "unit" measures in a set (e.g. X,

Y, Z) can thus be compared with one another by comparing the ordered series that each produces when it combines with the "unit" measures in the *other* set (A, B, C). Comparing the series, however, entails comparing the particular exponents that each series contains: so the qualitative difference between X, Y and Z resides in the series of particular differences between XA, YA and ZA, between XB, YB and ZB, and so on. All new measures in each series, such as XA and YB, thus have a dual status. On the one hand, each is one of the *exponents* of the series to which it belongs; on the other hand, each is also one of the *comparative numbers* (*Vergleichungszahlen*) in that series – that is, one of the numbers in terms of which that series is *compared* with another series in the same set.

It should also be noted that a particular *sequence of exponents* in the second set coincides with a *sequence of comparative numbers* in the first set, and vice versa. As Hegel himself puts it, these "exponents are the comparative numbers of the measures just named among themselves" (that is, the numbers *for* the comparison of such measures among themselves); so XA, YA and ZA form the series of exponents of *A*, but they are also numbers in terms of which *X*, *Y* and *Z* are compared. "Conversely, the comparative numbers of the independent measures among themselves, now also taken singly, are similarly the series of exponents for the members of the first series"; so XA, XB and XC are comparative numbers for *A*, *B*, *C*, but the exponents of *X* (SL 307 / LS 395).

Hegel ends 1.3.2.A.b.2 by turning his attention to the initial "unit" measures in each set of series – that is to say, their original *numbers* – rather than the series of exponents to which they give rise. The lines in which he comments on these numbers can, however, easily lead to confusion if read without care. The point to bear in mind, if confusion is to be avoided, is that the original *numbers* in each set form an ordered "series" (*Reihe*) – namely, X, Y, Z or A, B, C – just as the *exponents* produced by those numbers do. Indeed, it is precisely this fact that enables the exponents of two different numbers from the same set to be compared: as we saw above, the exponents of X and Y are comparable, only insofar as each combines with A, B and C *in that order*. Whereas, however, the different series of exponents within the same set are *comparable*, the two series of numbers that generate the two different sets – namely, X, Y, Z, on the one hand, and A, B, C, on the other – are described by Hegel as "opposite" (*gegenüberstehend*) series (SL 307 / LS 395). This is not to deny that one might be able to compare the exponents of, say, X and A, but in that case these two would have to form (or be part of) a new series of numbers, which would produce exponents by combining with a different, "opposed" series of numbers (say, Y, B, P and Q).

In order to make what follows as unambiguous as possible, I shall now distinguish a *series* of exponents from a *Series* of numbers by capitalizing the latter.[34] Note that the logical structure of the real measure at this point requires

that there be (at least) *two* Series of numbers, not just one. In the first form of real measure, exemplified by specific gravity, two individual measures are compared with one another directly because they share a common unit (e.g. a volume of 1) (see SL 304 / LS 391). In the new, more complex form of real measure, two measures (X and Y) can be compared, and so stand in a definite ratio to one another, because they form comparable series of exponents by combining with a common Series of different measures (A, B, C) that have their own numerical values. These latter measures in turn must be able to form their own series of exponents by combining with a Series that includes the first two measures (X, Y, Z). As we have seen, therefore, a real measure that relates the exponents of one measure (X) to those of another measure (Y) also makes it necessary that there be two different *sets* of measures generated by two opposed *Series* of measures. Moreover, as just noted above, the order of a number's exponents in one set – such as XA, YA and ZA – coincides with an ordered sequence of "comparative numbers" in the other set, and vice versa. The consequence, Hegel claims, is that the original numbers or quanta in each opposed Series actually perform a *threefold* function.

First, he writes, "each number" in a Series "is simply a *unit* with respect to the opposite Series in which [*an der*]" – that is, in combination with which – "it has its being-determinate-for-itself [*Fürsichbestimmtsein*]" – its distinctive quality – "as a series of exponents" (SL 307 / LS 395, emphasis added). So the number X is the unit that forms a distinctive series of exponents with A, B and C, and the same is true of Y and Z.

Second, however, each number is "itself one of the *exponents* for each member of the opposite Series" (emphasis added): so X, Y and Z are the exponents of the three series produced by A, B and C. Strictly speaking, of course, this is not true, since each such exponent is the combination of a number from one Series with a number from the other Series. Yet, if the number to which a series of exponents belongs is regarded as the unit that is common to each exponent, then what makes the difference between the exponents are obviously the other numbers with which it is combined. So in the series of exponents of the unit A, namely XA – YA – ZA, it is the different numbers, X, Y and Z, that distinguish the exponents from one another (even though X and Y are not themselves the same amounts as XA and YA). In that sense, Hegel can say that each number is *itself* an exponent for the numbers in the opposite Series.

Third, Hegel writes, each number is "a *comparative number* in relation to [*zu*] the rest of the numbers of *its* Series" (emphasis added): X is thus a comparative number in relation to Y and Z. Once again, this is not, strictly speaking, true, since the comparative numbers that distinguish the series generated by X, Y and Z are not just X, Y, Z themselves, but XA, YA and ZA, then XB, YB and ZB, and so on. X can, indeed, be considered to be a

comparative number in its own set, insofar as it is what distinguishes XA within one series of exponents from YA and ZA in the two other series in that set. It is important to note, however, that X and Y are not comparative numbers *all by themselves*. They can be regarded as comparative numbers, *only* insofar as they make the difference between XA and YA, or between XB and YB, that is, *only* insofar as first A and then B is common to them. Numbers, such as X and Y, thus act as comparative numbers, distinguishing two series of exponents from one another within a set, only insofar as they are exponents of those series and share a "unit" that is itself a number from the "opposite" Series (namely, A, B or C). This, I think, is what Hegel has in mind in the last three lines of 1.3.2.A.b.2, when he states that a number acts as comparative when, "as such an amount, which belongs to it also as an exponent, it has its unit, determinate for itself, [*ihre für-sich-bestimmte Einheit*] in the opposite Series".[35]

MEASURES AS EXCLUSIVE UNITIES

In the sphere of quantity a degree is a quantum that acquires its determinacy through the quanta that lie outside it. The 3rd degree is the 3rd, not because it contains three units within itself, but because it is preceded in a scale by two other quanta (see SL 183-4 / LS 232-3). The independent measures we have just been considering can also be said to be "self-external" in a certain respect, since they exhibit their determinacy *outside* themselves. In this case, however, that "outside" comprises, not just a set of quanta, but "a series of ratio-numbers" (*Reihe von Verhältniszahlen*), or exponents, that are produced when a measure combines with other measures (SL 307 / LS 395).[36] A measure is external to itself, therefore, because it exhibits its specific quality, or "being-determinate-for-itself", most fully in the distinctive ways it relates to and unites with what is *other* than it: "*its relating to the other* becomes that which constitutes the specific determination of this independent measure" (SL 307 / LS 395). The measures we are looking at differ from degrees, since a degree is determined wholly by its place in the scale to which it belongs and has no separate numerical value of its own, whereas these measures are independent ratios with their own original exponents. Nonetheless, like degrees, such measures are profoundly dependent on *others* of their kind: for each measure displays its independent character in the combinations it forms with other measures, and the way in which it combines with another to produce a new exponent "is determined just as much by the other as by the measure itself" (SL 308 / LS 395).

Hegel now points out that there is a further level of logical complexity to be uncovered in the exponents that are generated by combining measures. When two measures are combined into one, each is changed into the new measure and in that sense is "negated". Each is also negated, insofar as it is "specified" by the other – a reciprocal specification that is evident in the change in volume when

two densities are combined. "There is not, therefore, only *one* negation" – since one is not simply absorbed into the other – but "in the relation the two measures are *both* posited as negative" (SL 308 / LS 395). Yet in being negated in the new unity, each measure also "preserves itself in it indifferently" insofar as its quantum continues into that unity: in the case of densities each contributes its unchanging mass to the new combined measure. Furthermore, each measure specifies the other through being an *independent* measure. So although each is specified *by* the other and in that sense loses its independent identity, the fact that it specifies the other in turn shows that it does not lose that identity altogether. To put this another way, neither measure is simply *negated* when combined with another, but "*its negation* is in turn also *negated*".

Each measure, in combining with the other, thus proves to be the "negation of negation", or an affirmative measure that, to an extent, retains a character of its own in being negated. As combined, however, the two measures are no longer wholly independent, since they are moments of the *unity* they form together. This unity, Hegel explains, is also the "negation of negation", but it is not in itself a moment of any further unity and so has an unqualified independence that its component measures do not enjoy. This unity, as we have just seen, is one in which each measure is specified or negated, so it is itself the *negation* of the two measures. Yet it is not just their negation, since each (to an extent) preserves itself and continues in it; it is thus, in this respect, the *affirmative* unity of the two measures. It is their affirmative unity, however, in *not* being their mere negation, and so in being the "negation of negation" (SL 308 / LS 396). That is to say, it is affirmative in being self-negating, and thus self-relating, negation.

This moment of *self-relation* makes the unity something of its *own*, something independent. Moreover, as just noted, its independence is unqualified (since it is not itself part of a greater unity). The unity of the two measures is thus a *being for itself* – an independent measure in its own right with its own distinctive quantum. Yet, as we have also just seen, this new measure contains within it the explicit moment of *negation*, and in this respect it exhibits the logical structure, not just of being-for-self, but more specifically of the *one* (*Eins*). The pure one, we recall, includes the explicit moment of negation in the form of a limit, and so is "the wholly abstract limit of itself" (SL 132 / LS 166). Accordingly, it is not just neutrally itself, but in being itself it explicitly excludes all other ones from itself (SL 138 / LS 174). Similarly, Hegel maintains, if the new measure is to be both radically *independent* and overtly *negative* – and to be both at once, as a unity – it too must not just be separate from, and indifferent to, the other measures in its series, but must clearly *exclude* the latter: for it must be explicitly negative *in* being independent of them. As Hegel puts it, therefore, when two measures, such as densities, combine into one, "this, their qualitative unity, is thus an *exclusive* unity existing for itself" (SL 308 / LS 396).

The measures and exponents in a series, generated by an independent measure X, are first of all just quanta that differ from, and so relate to, other exponents in their series and in other series from the same set. In this latter respect, as we have seen, they are the "comparative numbers" that enable the series to which they belong to be distinguished from another series. Now, however, these exponents have proven to be more than just quanta in relation to other quanta; they have proven to be exclusive quanta with a character and quality that is uniquely their own. Note that, in Hegel's view, it is only through being *exclusive* that the exponents prove to be *qualitatively* different: as he puts it, "it is only in this moment of exclusion that the exponents [. . .] have in them their truly specific determinacy in relation to one another, and their difference thus becomes at the same time *qualitative* in nature" (SL 308 / LS 396, emphasis added). Moreover, *as* quanta that differ qualitatively from one another, each exponent is a fully explicit *measure* in its own right. When an independent measure combines with other measures, therefore, what results is not just a series of measures in which that independent measure reveals its distinctive character, but a series of measures that are themselves wholly *independent*. The real measure has thus once again subtly changed its logical structure. It started out as the *relation* between independent measures and then mutated into the explicit *unity* of these measures (see 2: 284-5); it then mutated further into a *series* of such unities (see 2: 289-90); now it proves to be a series of such unities that are themselves independent – indeed, exclusive – measures.

These measures are exclusive unities, as we have just seen, because they are not mere quanta, but self-relating negations that are both wholly independent and explicitly negative, and so set themselves quite apart from one another. Their determinate character, of course, stems from the two measures that are combined in them. Yet the mere fact that the exponents of X result from combining the latter with A, B and then C does not by itself give each one an exclusive character, since they all have X in common. What gives each one its exclusive character is the fact that the measures it contains modify or specify (and so negate) one another in a combination that is *unique* to that exponent. X, A, B and C all have their own characteristic – and constant – ways of specifying other measures, ways that they bring to each of the exponents they produce; yet they are in turn specified in different ways *by* each of those other measures. So even though X is common to all its exponents, it appears – or can appear – in each in a different way, depending on the other measure that specifies it. Each exponent is thus a unique combination of two measures with distinctive ways of specifying other measures, and it is this unique combination that gives the exponent its specific and exclusive character.

It should be noted that Hegel does not himself explain in the way I have just done in the last paragraph what it means for combinations of measures, such as densities, to be exclusive unities. The explanation I have given, however, seems

to me to make good sense of such unities and to be consistent with the logic of real measure as Hegel has presented it so far.

ELECTIVE AFFINITIES

The idea that two measures form an exclusive unity implicitly contains a new idea that must now be rendered explicit. Insofar as X and A combine to form XA, they can be said to have a certain "affinity" (*Verwandtschaft*) for one another. Yet when XA, XB and XC are conceived as *fully explicit* exclusive unities, the relation between their component measures – between X and A, X and B, and so on – proves to be subtly different from what we have encountered up till now. As Hegel puts it, "the *affinity* of an independent measure with the several measures of the other side is no longer an indifferent relation but an *elective affinity* [*Wahlverwandtschaft*]" (SL 308 / LS 396). That is to say, an independent measure is no longer drawn "indifferently" to all the other measures with which it has an affinity, but it is now drawn to some to the exclusion of others and so, as it were, "chooses" the former over the latter.

Hegel does not himself explain the precise logical connection between the idea of an exclusive unity and that of an elective affinity, but the connection is fairly easy to see. Before we say more about it, however, one important point needs to be made. The seamless course of Hegel's argument in 1.3.2.A.b.3 makes it look as if simple *densities* prove to have an elective affinity for one another; yet Hegel goes on to distinguish between mere densities and chemicals, such as acids and alkalis (or bases), and makes it clear that only the latter exhibit such affinities (see SL 308-9, 317-18 / LS 396-7, 409).[37] Furthermore, at the end of the remark that follows 1.3.2.A.c Hegel states explicitly that "mixtures" (*Gemische*) of densities are not *exclusive* unities in the strong sense. He notes that the exponents of such mixtures are "incommensurable" with the ratios between the densities before they are combined, since "the volume of a mixture [. . .] is not of the same magnitude as the sum of the volumes of the materials prior to the mixture". Yet he insists that such exponents are nonetheless "not exclusive determinations of measure" but form a "continuous" progression (SL 318 / LS 410). X can thus combine with A, B, C and so on (provided the conditions are right), and does not form an exclusive bond with just one of them.

This conception of the combinations formed by densities might appear to be at odds with what Hegel says about them in 1.3.2.A.b.3. The appearance of contradiction can be removed, however, if we understand his claim in that paragraph to be that combinations of densities are, indeed, exclusive (in the manner explained at the end of the last section), but not exclusive in a *fully* explicit manner.[38] We can then understand elective affinities to arise only when what is implicit in the idea of an exclusive unity is rendered fully explicit. Simple densities, therefore, do not have an elective affinity for one another, but

such affinities are exhibited by *chemicals*, such as acids and alkalis, that have densities without being primarily defined by them.[39] Hegel does not himself draw this distinction between being (what we might call) incompletely exclusive and being exclusive in a fully explicit manner, but the distinction enables us to make sense of all he says about "mixtures" and "exclusive" unities, so I will assume here that he has it in mind (and, indeed, that it is required by the logic of the argument).

What, though, connects the idea of a fully explicit exclusive unity with that of elective affinity? This becomes clear when we compare such a unity in more detail with the unity produced by mere densities. Densities, as we have seen, combine with one another, and in so doing form a series of new measures with their own distinctive exponents. These measures are "exclusive" in the sense that the component measures in each one specify and modify one another in a unique way; but they are not exclusive in a fully explicit manner because they do not exclude one another *altogether*. The density XA is quite different from XB, but it is not explicitly incompatible with the latter; accordingly, X, as a mere density, can combine with B, just as easily as with A (provided that the natural conditions, which are not a direct concern of logic, are present). Furthermore, XA and XB can themselves form a new unity together, which in turn has its own unique exponent. Indeed, X can combine with both A and B from the start (again if the conditions permit this): add the densities $\frac{2}{1}$, $\frac{4}{1}$ and $\frac{9}{1}$, and you get a new density of $\frac{15}{(3-v)}$.

By contrast, measure-combinations that are explicitly exclusive in the full sense *are* incompatible: as truly exclusive, XA and XB are not just different from one another, but one shuts out the other altogether. This does not mean that X, as the component of a fully explicit exclusive unity, cannot bond with both A *and* B, but it can do so only when it relates separately to each. When X is faced with A and B together, however, XA and XB reveal their incompatibility. In this case, since XA and XB exclude one another, X must bond with either A *or* B, and so must, as it were, "choose" between them. The fact that XA and XB are themselves mutually exclusive measures (rather than just unique ones) thus makes it necessary logically that X have an *elective* affinity for A as opposed to B (or vice versa). As Burbidge notes, such affinity shows itself "when two neutral salts are combined in a solution of water and exchange their radicals, producing two quite different salts. The fact that the acid radical of one salt abandons its base radical to take up with another from another salt [. . .] earned it the name of 'elective affinity'".[40] In this case the two acid radicals do not both combine with both base radicals, but each of the former "elects" to bond with one of the latter to the exclusion of the other. (Such reactions are now known as "displacement" reactions.)[41]

As we shall see below, a measure, such as an acid or base, remains, like densities, related to a set of other measures. It thus exhibits, in fact, not just one

elective affinity, or "preference" (*Vorzug*), for A over B, or B over A, but a *series* of preferences for A over B, B over C, C over D, and so on. Indeed, it is precisely in such a series that a measure, such as an acid, manifests its distinctive quality or "independent" character. In Hegel's words, the "independence" of a measure "shows itself in the fact that the affinities [of the measure] relate to one another in an exclusive way and that one has preference over another" (SL 311 / LS 399).[42]

THE "MORE OR LESS"

It is now clear why the idea of a fully explicit exclusive unity makes it necessary for its component measures to have an elective affinity for one another. That idea also makes it necessary for those component measures to stand in an explicitly *qualitative* relation to one another. Indeed, an elective affinity is itself precisely such a qualitative relation: it is an affinity between the qualities of two measures that shuts out another possible bond between measures.

The quality of a density consists initially in the *quantitative* ratio that constitutes it and distinguishes it from another density.[43] When two densities combine, they then exhibit a more overtly *qualitative* character by specifying one another in distinctive ways.[44] Their combinations themselves then prove to be qualitative, insofar as they are exclusive unities (see 2: 298-9). Combinations that are exclusive in a fully explicit sense, however, prove to be qualitative in a fully explicit manner, and so their components must themselves be explicitly qualitative. Chemicals, such as acids and alkalis, are thus distinguished not just by their density, but also by their distinctive *quality*, or chemical character. The elective affinity that exists between such measures is in turn an attraction that their qualities have for one another and so is an "exclusive, qualitative relation" (SL 309 / LS 397).

Yet measures, such as chemicals, are also quanta, and so combine in certain quantities, even though they have qualitative, elective affinities for one another. They thus stand in both a qualitative *and* a quantitative relation to one another. Moreover, Hegel points out, insofar as such measures are related as *quanta*, they exhibit a certain "indifference" to one another after all, since quanta, as we know, can be combined equally well with a whole range of other quanta. Accordingly, an independent measure with an elective affinity for another measure does not relate to, and combine with, just that other, but "relates to a *plurality* [*Mehreren*] of its qualitatively other side" in many different processes. Like mere densities, chemicals can form unities with a "Series of different others [*Differenten*] opposed to it" (SL 308 / LS 396).

Yet this latter idea is itself to be taken in *two* distinct senses that reflect the fact that the measures we are considering are explicitly qualitative *and* quantitative. On the one hand, a measure displays elective affinities with a

range of other measures, and thereby exhibits its distinctive *quality* in a series of preferences. On the other hand, however, that measure, as a quantum, must also have a *quantitative* relation to those other measures that differs from its explicitly qualitative relation to them. We have considered elective affinities as such above, so we now need to examine the distinctively quantitative relation that the current measures have to one another. This relation is governed by what Hegel calls "the *more* or *less*" (SL 309-10 / LS 398). Unfortunately, Hegel does not explain why this latter idea is connected to fully explicit exclusiveness, but it is possible to reconstruct the logic connecting the two.

The lines that help us do so are to be found in the last paragraph of 1.3.2.A.c. In this passage Hegel first states that when measures, such as chemicals, form unities, "the exponent" of each unity "is essentially a determination of measure and thereby exclusive". He then notes that "in this aspect of exclusive relating, the numbers" – of the various exponents – "have lost their continuity and their ability to flow together [*Zusammenfließbarkeit*]". The numbers lose their *continuity* with one another precisely because the unities to which they belong explicitly *exclude* one another and thereby prove to be qualitatively distinct. As we have seen, this occurs when X "chooses" to bond with A to the exclusion of B, or with B to the exclusion of C, and so XA excludes XB, and XB excludes XC. The exponents thus lose their continuity when the unities to which they belong are formed by *qualitative* "elective" affinities between their component measures.

As we have noted, however, the components of those discontinuous unities with their discontinuous exponents also stand in a *quantitative* relation to one another. That quantitative relation must, therefore, be compatible with, and indeed in some sense underlie, the discontinuity between the unities. This is not to deny that there must be a difference between the quantitative and the qualitative relation between the measures that form such unities; nor is it to deny that only the qualitative "elective" affinity between measures can ground the sharp, qualitative discontinuity between the unities they form: only the fact that X combines with A to the exclusion of B explains why XA explicitly excludes, and so in that sense is discontinuous with, XB. Nonetheless, the *quantitative* relation between X and A, and X and B, must also be such that it grounds a *discontinuity* between XA and XB. This quantitative discontinuity, which falls short of sharp, qualitative, truly exclusive discontinuity, must be grounded, as we shall now see, in the difference of "more" and "less".

Both densities and chemical measures form unities; those formed by the latter, however, are no longer continuous with one another, whereas this is not true of those formed by the former. Recall that when one density, X, combines with a set of other densities, A, B, C, to produce a series of exponents, the two densities in each exponent are both modified or negated. This is what enables them to form a unique combination, that is, one that is exclusive, though incompletely so (see 2: 300-1). Yet each exponent in the series has X as its

common element or "unit". X is, indeed, modified in different ways by the different measures with which it combines, but it modifies each of them in turn in the specific way that is characteristic of it. This characteristic way of modifying, or "specifying", other measures establishes a continuity between the exponents of X.[45]

Now, however, the exponents of combinations, as exclusive, can no longer form a continuity in this manner. This in turn means that a measure, X, must now relate in a *different* way to each of the measures with which it combines. Yet it is not enough to say simply that X must modify *A*, *B* and *C* in different ways, for in this case there would still be a certain continuity in the way X relates to those other measures: A, B and C are measures with fixed numerical values and, even if X were to modify each in a different way, it would still relate in the same way to the simple *numerical* value of each. If this residual *quantitative* continuity is to be interrupted, too, then X must in fact relate to *different amounts* or *proportions* of A, B and C; that is, it must relate to more of A than of B, or more of B than of C, and so on. In this way, the continuity between the exponents of X is more clearly broken off: for X modifies A, B and C in different ways by combining with *more* of one and *less* of another. It is this "more or less", therefore, that confers an exclusive *quantitative* character on an exponent such as XA – a character that belongs to it alone and is clearly not that of XB or XC (but that falls short of being genuinely qualitative).

As already indicated, Hegel takes this way of relating to others to be exemplified by measures that are no longer defined just by their density but that have a distinctive chemical identity. In particular, he maintains, it is exhibited by acids and alkalis (or bases). Acids, on the one hand, and alkalis, on the other, are qualitatively different from one another. Yet their distinctive character resides not just in their "immediate quality", but also in the distinctive way they *relate to*, and combine with, one another. Moreover, it manifests itself in the "*quantitative* manner of the relating" – the fact that an acid is "saturated" by *more* of one alkali than another alkali, and that "one acid, for example, requires more of an alkali in order to achieve saturation with it than another acid does" (SL 310-11 / LS 399).

The particular amount of an alkali or base that saturates a given acid yields a "*measure of saturation*" for the acid (SL 310 / LS 399). Yet precisely because acids are quantitatively, as well as qualitatively, distinct from bases, each acid can combine its quantum with those of several different bases to form a series of different "neutral" compounds, and in each case, the acid will have a different measure of saturation. So a given unit of an acid might be saturated by 3 units of one base but by 4 comparable units of another base. Each measure of saturation is thus a different ratio between a *unit* of the acid and a particular *amount* of units of a base, and as such each has its own exponent. The distinctive character or quality of an acid will, therefore, be displayed in the series of

exponents that it generates with different bases. As Hegel writes, "this quantity determination in regard to saturation" – the amount of another matter required for saturation – "constitutes the qualitative nature of a matter; it makes it what it is for itself, and the number that expresses this is essentially one of several exponents for an opposite unit" (SL 310 / LS 399).

Like simple densities, therefore, acids and bases exhibit their distinctive quality in a *series* of combinations that have their respective exponents. In the case of the acids and bases, however, these exponents have the character of *discontinuous* unities, since they are generated by combining X, not just with the fixed values of A, B and C (such as their densities), but with *more* of one and *less* of another.[46] Moreover, two different acids, X and Y, do not combine with the same values of A, B, C (as two densities do), but one acid requires more of the latter in order to be saturated than the other does.[47]

A chemical measure X, therefore, displays its distinctive quality in *two* different ways: once in the series of quantitative combinations that it forms with different proportions – with *more* or *less* – of A, B and C, and once in the clear qualitative preference, or elective affinity, it has for A over, and to the exclusion of, B, and for B over, and to the exclusion of, C. Since X relates to the same measures – A, B and C – in each case, we might expect the two ways in which a measure displays its quality to coincide with one another. In Hegel's view, however, things are not quite that simple.

As Hegel points out, the chemist, Jeremias Benjamin Richter, first established in the 1790s the determinate *proportions*, or what are known as the "stoichiometric" ratios, in which acids and bases saturate one another (SL 314 / LS 404).[48] Richter's results were then summarized "in their simplicity" by Ernst Gottfried Fischer in a table appended to his translation of Berthollet's essay on the laws of affinity in chemistry, published in 1802 (SL 311 / LS 400). According to this table, 1,000 parts of sulphuric acid are neutralized by 672 parts of ammonia, 793 parts of lime and 859 parts of soda, whereas 1480 parts of acetic acid and 1583 parts of citric acid are neutralized by these amounts of the bases.[49] Fischer's table thus clearly shows the distinctive series of stoichiometric ratios for each acid and base.

Hegel contends, however, that the series of preferences, or elective affinities, exhibited by acids and bases do not simply coincide with their respective series of stoichiometric ratios.[50] (In other words, the strength of an acid's elective *affinity* for a base is not simply a function of the *quantity* of the base needed to saturate a given amount of the acid.) Hegel makes this point in the remark on chemistry in the *Logic* and in his 1819-20 lectures on the philosophy of nature. In the latter he states that stoichiometric ratios "give the quantities with which bases and acids neutralize or saturate one another, but not the elective affinity. The series in which one base drives out the other did not match the stoichiometric ones" (NP 100-1). In the *Logic* (1832) he then repeats this view by writing that

"there is a well-established and precise difference between chemical affinity in a series of quantitative ratios and elective affinity as an emerging qualitative determinacy, the behavior of which in no way coincides [*keineswegs zusammenfällt*] with the order of that series" (SL 315 / LS 405). Hegel is a philosopher, not a chemist, but his contention is supported by others with more detailed knowledge of chemistry and its history. Ruschig, for example, also insists that elective affinities and stoichiometric ratios "stand in no systematic relation with one another", and that one cannot deduce the former directly from the latter.[51] As Ruschig explains:

> Elective affinity is specifically exclusive. Example: sulphuric acid drives out [*verdrängt*] hydrochloric acid from common salt [sodium chloride]; hydrochloric acid, however, cannot drive out sulphuric acid from Glauber's salt [sodium sulphate decahydrate]. This specific exclusion cannot be brought into a directly or inversely proportional connection with the (quantitative) relation between the equivalent weights of sulphuric acid and hydrochloric acid (49 : 36).[52]

The exclusion is not "proportional" with the quantitative relation between the acids because it cannot be explained directly by that relation alone. What is it about this quantitative ratio, as opposed to another (say, 59 : 26), that would generate an exclusion? This cannot be seen purely from the numbers, which lead one to expect only that the two acids will both combine with sodium in different proportions.

As the lines above from the *Logic* indicate, the reason, from Hegel's point of view, why the order of elective affinities between measures does not simply coincide with the order of stoichiometric ratios between them is that elective affinity is a *qualitative* relation between measures, rather than a merely quantitative one. Of course, those ratios also express the distinctive qualities of the acids and bases concerned (see SL 310 / LS 399). They express those qualities, however, in explicitly quantitative terms: the fact that X combines with *more* of A than of B. An elective affinity, by contrast, manifests the qualities of a measure, without the "more or less", in a directly qualitative way: when X is faced with A and B together, it combines with A, *not B*. The sphere of quality, we recall, is that in which there is an immediate difference between being and *non*-being (despite their inseparability and unity), and this simple, immediate difference is essential to an elective affinity. Such affinities are thus explicitly qualitative relations between qualities, whereas stoichiometric ratios are not. For this reason the orders generated by the two kinds of relation do not simply coincide. The stoichiometric ratios between a given acid and a set of alkalis do not, therefore, show for which of the alkalis the acid has the strongest elective affinities, even though those

ratios must be respected in any combinations formed on the basis of such affinities.

ELECTIVE AFFINITY AND "INTENSITY"

Three things have thus now become clear about the measures with which we are concerned here (in 1.3.2.A.c). First, they relate to one another in two distinct ways, not just one: through the "more or less" and elective affinity. Second, each way manifests the distinctive qualities of the measures concerned (albeit in a quantitative manner in one case and an explicitly qualitative manner in the other). Third, both ways are made necessary by the fact that the unities, or compounds, produced by the measures are exclusive and discontinuous.

In the last two paragraphs of 1.3.2.A.c Hegel sheds further light on the nature of elective affinity by contrasting his conception of it with the idea that an affinity has a certain degree or "intensity". Hegel first reminds us that an elective affinity is an "exclusive, qualitative relation" that "escapes" quantitative difference: X simply "chooses" to unite with A, *rather than* B (SL 309 / LS 397). This in turn means that X can be said to have a "*firmer*" attachment to, or "closer" affinity with, A than B.[53] Hegel then notes, however, that it is initially tempting to base such an affinity precisely on the *quantitative* relation in which the measures stand to one another, that is, on their stoichiometric ratios. Acids and bases combine according to such ratios, and it is easy to think that the strength of the affinity between X and A and the relative weakness of the affinity between X and B is in each case a function of the amount of the one that is needed to saturate the other. In Hegel's words, "the initial [*nächste*] determination that offers itself" is that a difference in elective affinity corresponds to "the difference in amount, thus in *extensive* magnitude, that exists between the members of the one side for the neutralizing of a member of the other side" (SL 309 / LS 397).

There are, however, two problems with this thought. First, as Hegel contends (and as is confirmed by Ruschig), there is no strict correspondence between elective affinities and stoichiometric ratios.[54] Second, basing elective affinities on ratios between amounts, or quanta, actually deprives those affinities of their distinctively exclusive, qualitative character. To see this, however, we need to look closely at what such a move *would* involve.

Consider three measures, X, Y and Z, all of which can combine with a fourth measure, A, and assume that X has the strongest elective affinity for A, and Z the weakest. Now assume, too, that a unit of X combines with (or is "saturated" by) 3 units of A, whereas a unit of Y combines with 2 units of A, and a unit of Z with just 1. The ratio between X, Y and Z with respect to A is thus 3 : 2 : 1, and these numbers are in turn the exponents of the combinations XA, YA and ZA. Each number, of course, is an extensive magnitude or amount; but, as was demonstrated in the section on quantity, every extensive magnitude is also an

intensive one, or a degree. The exponents of XA, YA and ZA must, therefore, be ranked 3rd, 2nd and 1st on a descending scale (with 3rd the highest and 1st the lowest). Since we assume here that the strength of the elective affinity between measures is based on the ratios in which they combine, this scale can now be used to assign a *degree* to the strength of each of their combinations and thus to their affinities for one another. In other words, the "*firmer*" combination, XA, is "converted" (*umgewandelt*) into a combination of "greater *intensity*" than ZA (SL 309 / LS 397). More precisely, XA is seen, not just to be "firmer" or "stronger" than ZA, but to lie *two* places above it in the scale of strength (whereas TA lies another four places above XA.)

A degree, however, is an intensive magnitude with a definite number, or extensive magnitude, attached to it. We are thus able to assign a determinate *amount* of intensity to the strength of each combination or affinity and so to measure it even more precisely. To say simply that X forms a "firmer" union with A than Y does, and that Y forms a "firmer" union than Z does, is to leave the relative strengths of XA, YA and ZA undetermined. By contrast, to say that XA has three times the strength of ZA, and YA twice the strength, allows us to measure the relative strengths of those combinations, and so of the affinities of X, Y and Z for A, exactly.

Now as we have noted, the elective affinities of different measures cannot in fact be measured in this way, since such affinities and stoichiometric ratios "stand in no systematic relation with one another".[55] (The table of stoichiometric ratios between X and A, X and B, and so on, does not show the relative elective affinities that X has for A, B, C and so on.) Even if affinities could be measured in this way, however, this would not explain why they are *elective* affinities: for to know that X's affinity for, and union with, A is 3 times as strong as that of Z does not explain why XA *excludes* ZA (when all three measures are together). In Hegel's words, if we were to assign definite degrees or amounts of intensity to different measures and their combinations, "no exclusion would be posited" (SL 309 / LS 398).

On the hypothetical model we are considering, the different degrees of affinity exhibited by X, Y and Z for A simply reflect the ratios in which X, Y and Z combine with (or are "saturated" by) A. All that those degrees of affinity tell us, therefore, is that when X, Y, Z are together combined with A, they will, or should, combine with A *in different proportions*: so X will combine with 3A, Y with 2A and Z with A. This, however, leaves unexplained why XA should exclude ZA. It is possible, of course, that X will absorb all of A and in that sense "exclude" the possibility of other combinations. As Hegel puts it, however, "there could take place equally well just one combination, or a combination of an indefinite number of members, provided that the portions [*Portionen*] of them entering into the combination corresponded to the required quantum in accordance with the ratios between them" (SL 309 / LS 398).

It is tempting to base the relative strengths of elective affinities on the stoichiometric ratios in which measures combine, for to do so is to assign a definite degree or amount to each combination of measures and to make it *measurable*. Yet it is also to *reduce* that combination (and its strength) to a quantum that is just "more or less" than another. This in turn is to deprive it of its exclusive, *qualitative* character. If, however, combinations of chemical measures are no longer exclusive qualitative unities, then two things are lost. First, the affinity between X and A is no longer *elective*, since A no longer has to "choose" between X and Y, but it can combine with both, provided it does so in the right proportions. Second, the components of those combinations are no longer qualities either (but amounts that are "more" or "less"), so there is no longer any *qualitative* affinity between the measures concerned.

The thought that the preferences, or elective affinities, of measures are based directly on stoichiometric ratios, and that the combinations formed through such affinities have a measurable intensity, thus *dissolves the very idea of an elective affinity*. This idea, however, has been shown to be logically necessary. Accordingly, Hegel writes, "the combination" (of X and A, or of Y and A) "which we have also called neutralization, is *not just the form of intensity*" – or does not just *have* that form – but is qualitative and exclusive (SL 309 / LS 398, emphasis added). This is not to deny that a measure's exclusion of another in favour of a third can be understood comparatively as its "closer" or "greater" affinity for the latter.[56] Yet the strength of that affinity should not be understood quantitatively as a greater *degree* or "intensity" that is based on the relative *amount* needed to saturate the measure, that is, on a "more or less": for to do so is precisely to omit the idea of *exclusion*.

This, however, is not itself to deny that elective affinities produce combinations of *quanta*: chemical measures that exhibit such an affinity combine according to a definite quantitative ratio (and so do the sounds that produce distinctive harmonies). Yet these combinations are at the same time exclusive and negative, since they shut out other possible combinations, and this in turn makes them distinctive, qualitative combinations.[57] Accordingly, they are not just combinations of *quanta*, but fully explicit *measures* in their own right: quantitative unities that are also qualitative ones. As Hegel writes, "the exponent" of the combination "is essentially a determination of measure and thereby exclusive" (SL 309 / LS 398).

To repeat: measures that have an elective affinity for one another, X and A, combine in a certain ratio that is greater or smaller – "more or less" – than the ratio in which Y combines with A. The combination they form is, however, an exclusive one: XA excludes YA. In this case, therefore, as Hegel puts it, "the *more* or *less* [. . .] acquires a negative character": it characterizes a combination that itself excludes another one (SL 309-10 / LS 398). Yet that "more or less" *itself* – the quantitative ratio as such – does not explain why XA excludes YA,

rather than the other way around. Indeed, the fact that X is saturated by more (or less) of A than Y is does not explain why any *exclusion* occurs at all. What explains the exclusion can thus only be the *qualities* of the measures concerned: the qualities of X and A are such that they combine to the exclusion of YA. Those measures have, indeed, to combine in a certain quantitative ratio, but that ratio as such does not ground the exclusive character of the combination. It is just the ratio that is specific to the measures that have a qualitative, elective affinity for one another. The "preference" (or "advantage") (*Vorzug*) that one combination of measures (and its exponent) enjoys over another is thus not simply a quantitative one: it "does not remain confined to the quantitative determinacy". It is a qualitative preference that goes hand in hand with (but is not reducible to or derivable from) a quantitative ratio between measures.[58]

Now, as I argued above, the fact that measures combine with *more* or *less* of one another according to certain ratios introduces a certain quantitative discontinuity into the series of combinations produced by a given measure (and between the series produced by different measures): for X will combine, not just with A, B and C (as happens in the case of densities), but with 3A, 5B, 9C and so on (and Y will combine with 4A, 8B, 13C). Nonetheless, the exponents of those combinations can be said to be continuous since they can be arranged into a sequence that increases quantitatively. Insofar as those measures exhibit elective affinities, however, the combinations they produce are much more radically discontinuous, because they *exclude* other combinations, rather than just differing from them quantitatively. So although X (say, sodium) can combine with A and B (say, sulphuric acid and hydrochloric acid) in different proportions, XA *excludes* XB (in the sense that, as Ruschig puts it, "sulphuric acid drives out hydrochloric acid from common salt" by reacting with the sodium in the latter to produce sodium sulphate).[59] In Hegel's words, therefore, "in this aspect of exclusive relating, the numbers have lost their continuity and their ability to flow together" (SL 309 / LS 398).

Once again: insofar as they combine quanta, the combinations produced by the current measures are in one sense discontinuous, but they remain nonetheless relatively continuous; insofar as they are qualitative, exclusive unities, however, they are radically discontinuous. This unity of continuity and discontinuity will be a central feature of the next form of measure: the nodal line. Before we examine the latter, however, there is a complication in the current form of measure that needs to be considered.

COMPLICATIONS: HEGEL AND BERTHOLLET

The first measure we encounter in the *Logic* is a simple quantum, or range of quanta, that is attached to, and sustains, some distinct quality (such as the liquid character of water). The two sides of a realized measure – for example, $s = at^2$

– are also quanta of distinct qualities (such as space and time). The two sides of a *real* measure, however, are initially quantitative ratios, such as densities, and it is these ratios *alone* that give each side its "qualitative nature" (most clearly when each is compared explicitly with its counterpart or another ratio) (see SL 304 / LS 390). In this case, therefore, the immediate distinction between quality and quantity that belongs to the first measure disappears. When two measures form unities that are fully and explicitly exclusive, however, that distinction re-emerges for two reasons. First, as we have seen, the moment of explicit exclusion turns such unities into qualitative ones whose components are themselves explicitly qualitative, as well as quantitative (for example, chemicals rather than mere densities).[60] Second, a component, X, displays its quality in certain qualitative, elective affinities, in which it combines with A to the exclusion of B, or with B to the exclusion of C, but also in specific quantitative ratios that have to be observed in such elective combinations – the fact that it combines with more of A and less of B.

In the sphere of fully exclusive unities and elective affinities, therefore, there is a difference between quality and quantity. Yet – and this is an important *yet* – this difference is not absolute, for elective affinities obtain between measures, and a measure is the *unity* of quality and quantity. Indeed, this unity is so thoroughgoing that it leads to a subtle complication in the very idea of elective affinity.

Two measures may well have a qualitative affinity for one another; but each is at the same time a *quantum* and, as such, can combine equally well with a host of other measures that are also quanta. As Hegel puts it, it is thus "a matter of indifference to a moment" – to a measure – "that it receives its neutralizing quantum from several opposite moments" (SL 310 / LS 398). So far, however, there has been a clear distinction between the possibility of a measure combining with several other measures and its elective affinity for one over another: the former possibility, grounded in the fact that the measure is a quantum, depends on its encountering other measures separately – one after the other – whereas its qualitative elective affinity manifests itself when it encounters those measures together (2: 301). Yet this distinction cannot in fact be drawn so clearly: for measures with an elective affinity for one another are always, irreducibly, *quanta*, and as such must always be able to combine *with more than one other*. This in turn means that a measure can combine with two, or perhaps several, others in different proportions, not only when it encounters them separately, but also when it encounters them *together*. That is to say, it can behave as an indifferent quantum, not only when it does not, and cannot, display its qualitative, elective affinity, but also when it *is* supposed to do so.

What is it, therefore, that determines whether a measure combines with some or all of the others it encounters, or just one? This cannot simply be the qualitative affinity itself, since, as just noted, this cannot eliminate the possibility

that the measure will combine with more than one of the measures before it. What determines whether the measure combines with just one, or more than one, of the others must therefore be a *quantitative* feature of the measures concerned. Yet this in turn cannot simply be the *ratio* in which the first measure is "saturated" by the others, since, as we have seen, this ratio is precisely unable to explain why the measure should combine exclusively with one of its counterparts (2: 305-7). What determines whether a measure, X, combines with A alone, rather than with A and B, can thus only be the simple *quantity* in which all three are present, that is, the quantum of each measure: a measure shows an elective affinity for one among two or more other measures, when they are all present in the right amounts. This does not mean that one measure does not actually have any qualitative, elective affinity for another, but it does mean that the concept of such an affinity is partially undermined: for such a qualitative affinity can manifest itself only when the quantitative conditions are right. In Hegel's words, "the exclusive, negative relating [of the measure] at the same time suffers this impairment [*Eintrag*] from the quantitative side" (SL 310 / LS 398).

To repeat: Hegel's point at the end of 1.3.2.A.c is not that there are in fact no elective affinities, but only stoichiometric ratios – that all a measure ever does is combine with more of another and less of a third. The point he is making is that there are, certainly, elective affinities between measures, but that these are inseparable from, and dependent on, the quanta of the measures concerned. Thus, when faced with A and B together, X will combine with different proportions of both of them, unless the *quantities* are such as to allow it to display a *qualitative* preference for one and combine with the latter to the exclusion of the other.[61] Quantity thereby enables a qualitative relation to manifest itself, and so the latter remains entangled with the former even though it is also distinct from it. Accordingly, as Hegel puts it, "what is posited here is a conversion of an indifferent, merely quantitative relating into a qualitative one, and, conversely, a transition of a specific determinacy into a merely external relation" (SL 310 / LS 398).

This important modification of the idea of elective affinity is made necessary by the logic of measure itself. Measures are now no longer just quantitative ratios in relation to one another – like densities – but they also exhibit exclusive, qualitative preferences for one another. In this sense, a difference between quality and quantity re-emerges in the sphere of measure. Nonetheless, those qualitative preferences are thoroughly bound up with, and mediated by, the quantitative relation between the measures. This reflects the intimate unity of quality and quantity that constitutes the very idea of measure as such. In the long remark that follows 1.3.2.A.c, it becomes clear that this result of the logic of measure is also consistent with the empirical findings of the French chemist, Claude Louis Berthollet (1748-1822), as Hegel understands them.

As is well known, Hegel notes, "Berthollet modified the general conception [*Vorstellung*] of elective affinity through the concept of the efficacy *of a chemical mass*" (SL 311 / LS 400). This modification, he continues, has had no effect on the stoichiometric ratios identified by Richter, but it has "not only weakened, but rather removed [*aufgehoben*] the qualitative moment of exclusive elective affinity" (SL 312 / LS 401). It has not done so, however, in *all* circumstances.

If, Hegel explains, two acids react with a base, and there is enough of the one said to have the greater affinity with the base to saturate the latter, then "according to the conception [*Vorstellung*] of elective affinity, this is the only saturation that results; the other acid remains quite ineffective and is excluded from the neutral combination" (SL 312 / LS 401).[62] Berthollet argues, however, that in fact *both* acids will combine with the base in a ratio determined by their "chemical mass", that is, by the amount of each acid coupled with its "capacity for saturation" (the capacity that grounds its stoichiometric ratios).[63] In this case, since both acids combine with the base, neither is excluded by the latter and so it displays no *exclusive*, purely "elective", affinity for one of the acids: it simply combines with more of one and less of the other.

Hegel insists, however, that Berthollet does not completely eliminate the idea of elective affinity: for he shows that there are conditions under which the base will, indeed, react with, and so "prefer", only *one* of the acids. As Hegel puts it,

> Berthollet's investigations have identified the more detailed circumstances under which the efficacy of the chemical mass is nullified [*aufgehoben*] and one acid (the one with the stronger affinity) appears to drive out and to *exclude* the action of the other acid (with a weaker affinity), that is, appears to be active in the sense of elective affinity.
>
> —SL 312 / LS 401

Yet Berthollet shows that such elective affinity does not depend on the "qualitative *nature* of the agents as such" – that is, on that nature *alone*. "Exclusion takes place in certain *circumstances* [*Umstände*], such as strength of cohesion, or the insolubility in water of the salts formed", circumstances that can themselves be changed by other circumstances, such as temperature (SL 312 / LS 401).[64]

It is clear that Hegel accepts Berthollet's critique of *purely* elective affinity. In his view, however, the French chemist's findings do not completely destroy the idea of elective affinity, but rather make such affinity dependent on certain conditions. What he draws from Berthollet's investigations, therefore, is not the idea that elective affinity is a fiction, but that under a "particular set of circumstances, a thing *prefers* one combination to another".[65] These circumstances are *quantitative* and include the amounts of the chemicals

concerned (and so correspond to what is made necessary by the logic of measure); but they are also qualitative, since they include such features as cohesiveness and solubility. In both cases, however, when they are the right ones, they allow a chemical agent to display its qualitative, elective affinity for another. Thus, even though elective affinity is not rooted in the quality of the agent alone, but has further conditions, it remains the expression *of* that quality. Hence, Hegel states, "so far as the nature of the *qualitative* element present in elective affinity goes, it makes no difference whether this element appears in the form of those circumstances conditioning it, and is so interpreted" (SL 315 / LS 405).

It is evident from this remark on chemistry (and from Hegel's philosophy of nature) that Hegel acknowledges the considerable importance of Berthollet's work.[66] Yet it is not just the latter's empirical findings that challenge the conception that elective affinity is something unconditioned and invariant. This conception is also undermined by the speculative *concept* of elective affinity itself, which shows the exclusive affinity between the qualities of measures to suffer "impairment from the quantitative side" and so to have quantitative conditions (SL 310 / LS 398). Berthollet's empirical studies reveal that elective affinities between chemicals in nature have both quantitative and qualitative conditions; but the conception of an unconditioned elective affinity is already challenged and modified by the *logic* of measure itself.[67]

CHAPTER THIRTEEN

The Nodal Line and the Measureless

A measure, for Hegel, is not just something that can be "measured", something whose quantum can be determined. It is a quantity or quantitative relation that constitutes, or belongs to, the *quality* of something. The intrinsic logic of measure, so understood, requires it to undergo a distinctive development and so to take on various different forms. In its initial immediacy, described in 1.3.1.A, measure is simply a quantum (or range of quanta), standing alone, to which a given quality is attached. It then proves to be a measure in *relation* to another quantum, for which at first it provides a rule and which it then modifies or "specifies" (SL 291-3 / LS 375-7). After this, measure turns into a relation or ratio between two measures, each of which is a quantum of a given quality – a ratio that Hegel calls a "*realized*" measure (SL 295 / LS 380). The higher forms of realized measure are ratios involving quanta raised to a power, such as s^3 and t^2 – ratios that have the status of laws. We learn, however, that each such ratio also coincides with a direct ratio between simple quanta. The realized measure thus proves to be the unity of, but also relation between, *two ratios*, and thereby exhibits the form of a new measure that is no longer just a realized measure. When this form is rendered explicit and considered in its own right, it constitutes what Hegel calls a "real" measure (SL 302-3 / LS 388-9). In the latter's initial immediacy, described in 1.3.2.A.a, each of its constituent ratios is itself immediate. Accordingly, a real measure is at first a ratio between two direct ratios that are exemplified, in Hegel's view, by densities. A single density is a ratio between mass and volume, and a real measure is the ratio between one density and another; so, for example, the real measure or "specific gravity" of a metal is the direct ratio between the density of the latter and that of water (see SL 303-4 / LS 390-1).

The two ratios (or densities) in a real measure are then understood to form a *combination* together. In this way, measure again becomes, explicitly, a single measure standing alone, as it was at the start of its logical development.[1] Yet it is a single measure that results from combining *two* separate measures that are in a certain ratio to one another. Furthermore, each of these separate measures combines with a set of other measures and forms a *series* of unities with them. The real measure is thus not just a single measure after all, but a "series of measure-relations" (or measure-combinations) (SL 305 / LS 392). As we have just seen in the last chapter, these measure-relations, which are themselves combinations of measures, then mutate logically into explicitly *exclusive* unities (exemplified, in Hegel's view, by certain chemical compounds). Two such unities, XA and XB, are distinguished by the fact that the measure, X, combines with more of A than of B, and the fact that, under certain conditions, XA excludes XB altogether, just as XB excludes XC (SL 308-10 / LS 396-8).

When we look back over this logical development, we should bear two things in mind. First, a measure is something *for itself* and so exhibits the most developed form of qualitative being: being-for-self. Measure owes this characteristic to the logic of quantity. As we saw in 1.2.3.C, quantity culminates in a quantum that raises itself to a power of itself and so, in becoming another quantum, relates to *itself*. Such a quantum thus displays the quality of being-for-self, and when the fact that a quantum constitutes such quality is rendered fully explicit and it is conceived as wholly *self*-relating, without relating to another, the quantum is conceived as a measure (see 2: 207-8). Measure thus inherits its being-for-self from the logic of quantity that makes it necessary. Such being-for-self is in turn clearly manifest in the fact that the measure is initially a *single* measure standing on its own, that the two sides of a real measure are *independent* measures, and that the unities formed by those independent measures prove to be *exclusive* (and then explicitly exclusive) measures.[2]

Second, however, although measure is the unity of quantity and quality, it preserves the difference between the two – for example, in the difference between a rule and the quantum that it measures. The moment of difference in measure also finds expression in the fact that the latter proves to be a *ratio* between quanta of given qualities (in the realized measure) and then a ratio between two *ratios* (in the real measure). It finds expression as well in the idea of an exclusive measure: for being exclusively oneself consists not only in being purely *for oneself* – in being a separate, independent *one* (*Eins*) – but also in *differing* from – in shutting out – something else. In the explicitly exclusive unities formed by measures such as acids and bases, the two aspects of measure – being-for-self (or unity) and difference – are thus united, and exclusiveness proves to be the appropriate form for a real measure to take.

Hegel now examines more closely what is contained in the concept of an explicitly exclusive measure, as it has emerged at this point in the *Logic*. Such a

measure is a unity for itself, but it also contains difference. This difference itself takes two forms: the measure differs from other such measures, but it also preserves within itself the difference between quality and quantity. In this latter respect, the exclusive measure is thus, on the one hand, a *qualitative* unity and so combines two qualities; as such, it is the result of a special elective affinity between two measures. On the other hand, it is a *quantitative* unity and so combines two quanta of those qualities – quanta that can also combine with those of other measures. Hegel claims that this fusion of being-for-self and difference, and also of qualitative and quantitative unity, gives rise logically to what he calls a "nodal line of measure-relations" (SL 318 / LS 410). We must now consider why this should be.

THE NODAL LINE OF MEASURE-RELATIONS

Hegel begins his account of this nodal line by simply reminding us that "the last determination of the measure-relation was that, as specific, it is *exclusive*" (SL 318 / LS 410). This measure is exclusive, he notes, because it is a particular kind of unity, namely, a "*negative* unity of the distinct moments". In the sphere of quality the one (*Eins*) is simple and undifferentiated, and as such excludes other ones. By contrast, the explicitly exclusive measure we are now considering is not simple but the unity of two different measures, and it is only as such a unity that it is exclusive. It is exclusive, therefore, because those two measures form *this* unity or "neutrality" to the exclusion – and thus as the *negation* – of other possible unities. That is to say, this unity is exclusive because its components have a specific elective affinity for one another that, under the right conditions, prevents either from combining – at the same time – with any further measure.[3] These measures can, of course, also combine, separately or under different conditions, with different amounts – with "more or less" – of various other measures; but their unity is properly *exclusive* when it results from their elective affinity for one another. Accordingly, Hegel here equates the measure that is a "unity *for itself*" (fürsichseiende *Einheit*) with an "elective affinity".

Hegel then goes on to claim that "the *exclusive* elective affinity also *continues* into the neutralities that are other than it", that is, into other unities (SL 319 / LS 410). It is not immediately clear why he would make this claim: for surely, if a unity, or elective affinity, is exclusive, the one thing it does *not* do is continue beyond itself in other unities. Things become clearer, however, in the rest of the paragraph and the next.

Hegel notes first that, logically, an explicitly exclusive unity or "neutrality" remains "separable" into the two measures that combine to form it, since those measures enter into relation as "independent somethings" (SL 319 / LS 411). Densities, of course, also belong to things that are independent (2: 278); the

current measures, however, are more explicitly independent, since they are qualitatively distinct from one another and not just distinguished by their ratios, as densities are. "*Separability*" (*Trennbarkeit*) is thus inherent in particular in the exclusive unity as the latter is understood here. This unity is not just an entity in its own right but the unity *of* two independent measures that retain something of their independence in it; as such, it can be divided into them once again.[4]

These two measures, we recall, not only have a special affinity for one another, but, prior to forming a unity, they can also combine their quanta with those of *other* measures. As Hegel puts it, they are "indifferent [*gleichgültig*] to combining with this or with others of the opposite Series" (SL 319 / LS 411). As a quantum, each measure is thus in fact indifferent to the very measure with which it has a special, qualitative affinity, for the latter is itself just one among many with which it can combine. Furthermore, as quanta, the measures are inherently variable and can combine with other measures in different amounts: so one part of X might combine with two parts of A, three parts of X with five of B, and so on.[5]

Hegel maintains, however, that indifference and quantitative variability characterize the two measures not only *before* they combine to form an exclusive unity, but also *in* that unity itself. As he puts it, such a unity is "affected with an indifference of its own; it is in itself [*an ihm selbst*] something external and alterable in its relation to itself" (SL 319 / LS 411). This is because, as just noted, that unity remains *divisible* – divisible into two indifferent measures whose quanta can change. Yet there is a subtle ambiguity to this latter idea. On the one hand, it means that an exclusive unity can be *broken up* into its two independent measures, which are then free to form new and different combinations with other measures. On the other hand, however, it means that divisibility, indifference and quantitative variation belong *to* the exclusive unity itself. The first of these two thoughts takes us no further forward logically, but preserves the familiar idea that the measures can unite with *other* measures. The second, by contrast, does take us forward: for implicit in it is the thought that an exclusive unity is divisible and variable *in being* that very unity, not just because it can cease to be that unity (and break up altogether). When this thought is rendered explicit, a new form of measure arises: one that harbours divisibility, variability and indifference wholly *within* itself, that is, while *remaining what it is*.[6]

Since this new measure remains what it is, it continues to be the unity of its two constituent measures – to the exclusion of any others – whatever quantitative changes it undergoes. As this unity, it is, indeed, vulnerable to being divided into its constituents, which can then form new combinations that are "indifferent" to the way the latter are originally combined.[7] Yet, in being divided, the unity of the measures *remains* the exclusive unity that it is. This means that the constituents must once again combine *with one another*, not with any other measure.

There are, therefore, two sides to the new exclusive measure. On the one hand, it has a quantitative side that is changeable: so the measures that combine in it in these specific amounts can also combine with one another in different amounts. On the other hand, it has a qualitative side that is unchangeable and that gives the measure its exclusive character and being-for-self: it is the unity of *these* measures with *these* qualities to the exclusion of all other measures. This qualitative dimension thus remains constant as the quanta of the two measures vary. As Hegel writes:

> the *relation* of the measure-relation *to itself* [Beziehung *des Verhältnismaßes* auf sich] is distinct from its externality and variability, which belong to its quantitative side. As related to itself in contrast to these, it is an affirmative [*seiend*], qualitative foundation – a permanent, material substrate [*Substrat*].
>
> —SL 319 / LS 411[8]

We can now see, therefore, why Hegel states that "the *exclusive* elective affinity also *continues* into the neutralities that are other than it", that is, into other unities (SL 319 / LS 410). In saying this, he is rendering explicit what is logically implicit (but only implicit) in the idea of an exclusive affinity: namely that the measures that are bound together by such an affinity must be able to combine *with one another* in different amounts to form new and different unities or "neutralities".[9] In so doing, however, he is taking us from the thought of a simple elective affinity to that of an altogether *new* measure (or rather noting that this thought itself takes us to that new measure).

Now in the sphere of measure, as we have seen, a certain quantity is attached to a given quality, but this quality is itself understood to *specify* that quantity. This specification can occur in different ways, some of which are more active and transformative than others. In a simple, immediate measure, for example, the quality merely specifies a certain range of quanta within which the quality is preserved, whereas in a realized measure the quality requires its quantum to be raised to a power.[10] In an exclusive measure, as it appears in 1.3.2.A.c and the following remark, the qualities of its two constituent measures specify that their quanta be combined in certain proportions: so X must combine with more of A than B, and Y must combine with still different proportions of A and B. These stoichiometric proportions do not themselves determine the elective affinities of X, Y, A and B, but they must be respected in any combinations formed on the basis of such affinities (see 2: 306-7).

Since the new measure we are now considering is still an exclusive one, the qualities of its constituent measures still specify the proportions in which those measures must combine. This measure is inherently divisible and so can break apart, and its constituent measures can form new combinations in different

amounts. Indeed, since those constituent measures are quanta, they can increase or decrease and thereby form new combinations even without first being separated (in which case we might say that the forming of a new combination *itself* counts as the "decomposition" of the original one). This quantitative change is what Hegel refers to as the moment of "externality and variability" in the measure (SL 319 / LS 411). As the constituents change their quanta, however, the measure remains the exclusive unity of *these* two measures and no others: they form new combinations with different amounts *of one another*. Moreover, because the qualities involved do not alter, they continue to specify the quantities in which the measures can combine anew; that is to say, in Hegel's words, the qualitative "foundation" which is "the continuity of the measure with *itself* in its externality would have to contain in its quality the principle of the specification of this externality" (SL 319 / LS 411). So, the two measures in the current measure can combine in changing amounts, but however much of one measure there may be, the qualitative unity they form will continue to determine how much of the other measure will combine with the first.

As the quanta of the two measures change, however, their qualities must specify the varying quanta in *two* different ways. On the one hand, since those quanta form new combinations of the qualities, the latter must specify them in such a way that their combinations are, indeed, *new*. On the other hand, the qualities must also show that their varying quanta are a matter of *indifference* to them and to their exclusive unity.

First, insofar as such quanta are a matter of indifference, the qualities of the measures must, as it were, *allow* them to change without determining them anew. The qualities do, indeed, play a specifying role, but only insofar as they require the quanta to preserve the proportions in which they originally combine, *however* their actual amounts may change (to the extent, of course, that this is possible). In this way, a particular unity of X and A may change and become a new unity, as X and A increase or decrease, but the qualities of the two measures leave the ratio in which their quanta must combine unchanged. In this respect, those qualities remain *indifferent* to the fact that there is a new combination of quanta.[11]

Second, however, the qualities must also determine the quanta, in a more explicit and active manner, to be (or become) the quanta of a *new* combination. The qualities can do this, however, only by specifying anew what they control, namely the *ratio* in which the quanta stand to one another. X and A, combined in the measure XA, must, therefore, re-combine in a different ratio and in so doing form a new measure that can itself remain constant throughout further "indifferent" changes in the quanta concerned. The qualities of the measures in the current exclusive unity, which is itself a measure, thus require them, not only to increase or decrease in a fixed ratio to one another, but also to combine in new and different ratios and thereby to form new measures. These different

ratios constitute new measures by conferring new qualities on the combinations concerned; but the latter are nonetheless all produced by the *same* two constituent measures with the *same* qualities. The new measures are thus qualitatively distinct, but also different forms of *one* exclusive qualitative unity.

So, to recapitulate: the current exclusive measure, produced when two measures combine, remains inherently divisible, and those measures are themselves inherently changeable; yet throughout such change the measure also remains the unity of those two specific measures; the latter must, therefore, form new combinations *with one another* in different amounts. The qualities of the two measures, however, specify, on the one hand, that different combinations of them must preserve the same quantitative ratio as the original combination, but also, on the other hand, that some combinations must have new ratios and so constitute new measures.

The exclusive measure thus necessarily generates *new forms of itself*, some of which are distinguished simply by different quanta in relation to one another, and some by new ratios between quanta. In Hegel's own words, the measure is "external to itself in its being-for-self": it "posits itself both as another merely quantitative relation and as another relation which is as such at the same time another measure", and so is "a specifying unity which within itself [*an ihr*] produces measure-relations" (SL 319 / LS 411). This "self-externality" in turn explains why Hegel states on the previous page that the "*exclusive* elective affinity also *continues* into the neutralities that are other than it". It continues in this way because – in its current, more explicit form – it produces a series of further unities or "neutralities", all of which are different versions of itself (SL 319 / LS 410). When we first read these words of Hegel it is not obvious what he has in mind, but his meaning is, I hope, now clear.

Hegel notes that the various unities generated by the measures in the current exclusive measure are significantly different from the unities produced by independent measures such as densities. Both sets of unities form a series, but there the similarity ends. The series produced by a density X arises because the latter combines with a set (or Series) of *different* individual densities: so X gives rise to the series of unities, XA, XB, XC and so on. These unities, however, are all of the same kind: each is, in the same way, the combination of two densities or ratios, and they are distinguished only by the quanta involved in each case. By contrast, the unities produced by the current exclusive measure are all combinations of the same independent measures and so are all forms of the same qualitative unity (say, XA). As Hegel puts it, "they occur in *one and the same* substrate, within the same moments of the neutrality" (SL 319 / LS 411). The unities, however, are not all of the same kind, for some involve different quanta but preserve the same ratio between them, whereas others involve quanta in different ratios (and so constitute new measures). In Hegel's words, "the measure, in repelling itself from itself, determines itself to be other

relations, only quantitatively different, which likewise form *affinities* and *measures*, *alternating* with those that remain only *quantitative differences*" (SL 319 / LS 411).[12] Densities thus combine to form a continuous series of unities that differ only quantitatively; by contrast, an exclusive measure in its current sense produces a series of quantitatively different unities that is punctuated by new measures with new qualities (albeit produced by the same two constituents). Accordingly, Hegel explains, these unities "form in this way a *nodal line* [*Knotenlinie*] of measures on a scale of more and less".

The new exclusive measure in 1.3.2.B also differs from the explicitly exclusive unities, formed by acids and bases, that are considered in 1.3.2.A.c and the following remark. These unities are exclusive in the sense that they displace (or can displace) one another. This occurs (in Burbidge's words) when "the acid radical of one salt abandons its base radical to take up with another from another salt": so when XB is added to YA, X bonds with A and so forms XA to the *exclusion* of XB.[13] The new measure (in 1.3.2.B) still differs from and "excludes" other measures, that is, other unities or compounds; it does so, however, not by actively displacing such unities, but by preserving *itself* – preserving the same combination of qualities – throughout the quantitative changes that *it* undergoes. The new measure is thus overtly *self*-relating, or *for itself*, rather than in competition with *other* combinations (though it is "external to itself in its being-for-self" insofar it produces a series of different forms of itself [SL 319 / LS 411]). This point is implicit in the emphasis Hegel places on "the *relation* of the measure-relation *to itself*" and in his assertion that the relation of the new measure to other "neutralities" is not one of "comparison" (*Vergleichung*). It is also explicit in the contrast he draws between combinations of "the *same* moments of the neutrality" and "affinities" – such as we encountered previously – "in which an independent measure relates to independent measures of another quality and to a series of these" (SL 319 / LS 411, emphasis added).[14] It is true that metals, for example, can form exclusive compounds in both senses: they can displace other metals from the aqueous solutions of their salts (depending on their chemical reactivity),[15] and, as we shall see below, they can also form different compounds with the same element (such as oxygen). What is important to note here, however, is that there is a significant *logical* difference between the two forms of explicitly exclusive measure.

It is worth adding that the nodal structure we have highlighted above also distinguishes the multiplicity generated by the new exclusive measure from that produced by the pure one and by the pure quantum. The one and the quantum, like the current measure, are both characterized by being-for-self, and both also "repel themselves" and so reproduce themselves outside and beyond themselves. In so doing, the pure one gives rise to a multiplicity of simple empty ones, all of which are exactly alike (SL 135-6 / LS 171-2). The quantum, on the other hand, gives rise to a multiplicity of quanta, each of which is a *different* number;

yet all are equally extensive magnitudes and so in that sense they are, once again, all the same (SL 189-90 / LS 239-40). The unities produced by the current exclusive measure, however, are *not* all the same, even though they unite the same measures. These unities are not all qualitative units (like ones), nor are they all quantitative units (like numbers), but some are merely new *quantities* of the measures concerned, whereas others are different *measures*, whose distinctive ratios confer new *qualities* on them. This, indeed, is why the unities produced by the current exclusive measure form a *nodal* line: nodes are generated because the unities differ quantitatively, but in some cases also qualitatively, from one another. That is to say, they are generated by the unity of and *difference between* quantity and quality that is characteristic of measure.[16]

JUMPS IN NATURE (AND EXAMPLES)

We have said that the qualities of the two measures in the current exclusive unity specify the quantities of those measures that can (and must) be combined in new forms of that unity. This claim, however, needs to be understood with precision. The qualities determine the initial *ratio* in which the two measures must combine and which must initially be preserved as the amount of each is increased. They also determine new and different ratios in which the measures can be combined. They do not, however, determine the actual amounts there must be of each measure in a given case. So the qualities of X and A determine that so many parts of the former combine with so many parts of the latter (say, 2X and 3A), but they do not require there to be a particular *amount* of either one (2X and 3A, 4X and 6A, or 6X and 9A).

Since the particular amounts of the two measures in the current exclusive unity are not governed by their qualities, the latter are indifferent to the changes those amounts may undergo, to their increase or decrease. Yet there are limits to such indifference. A limit to quantitative change is, of course, already set by the simple "specific quantum" of a quality. This is the quantum beyond which a thing no longer has the quality concerned and so (when the quality defines the thing) is no longer that thing – for example, the size of population beyond which a state can no longer preserve its constitution (see SL 323 / LS 416). The limits set by the current exclusive measure, however, are more complex than this. As we have seen, they come in the form of ratios.

So once again: the nature or quality of X does not require there to be 100 units, rather than 200, of it – except when 100 is the "specific quantum" of X (and so the condition of its being X at all) – but this quality does determine that 100 units of X can initially combine only with a certain amount of A; that is, it determines the ratio that must first be preserved between the two measures (or, rather, this ratio is co-determined by X and A). If, therefore, an amount of X is combined with too much of A, then some of the latter will not be taken up by

X, but will be left as a residue. The qualities of X and A then also determine them to stand in further *different* ratios to one another, ratios that constitute different measures. If, therefore, the amounts of X and A in a measure increase or decrease to the point at which they stand in one of these new ratios, we will be confronted by a new measure and compound, not just by a different amount of the previous one. Not every change in the quanta of a measure's components produces a new measure in this way; in most cases changes simply give rise to a larger or smaller amount of the measure concerned. In some cases, however, changes will set the two components in a new ratio to one another that confers a new character or quality on the combination they form. The new quality is brought about by the changes in quantity; yet the emergence of that quality is ultimately determined by the *qualities* of the two component measures, since equivalent quantitative changes in two different measures might not bring them in the same way to a nodal point at which their combination is a qualitatively different measure.

Hegel sets out the general structure of such nodal change in the fourth paragraph of 1.3.2.B. "There is present", he writes, "a measure-relation, an independent reality that is qualitatively distinguished from others" (SL 319 / LS 411). This measure is the enduring exclusive unity of, and relation between, two measures that are themselves independent. Since the latter both have different quanta, the exclusive measure they form is "open to externality and the change of quantum". Furthermore, "it has a range [*Weite*] within which it remains indifferent to this change and does not change its quality" (SL 319 / LS 412). There comes a point, however, at which the change in its quantity does lead to a change in its quality: "the altered quantitative relation is converted into a measure, and thereby into a new quality, a new something" (SL 320 / LS 412).[17] Hegel reminds us that, as this new measure arises, the qualities of the two measures that together form the first measure do not themselves change: as he puts it, the new measure is determined by the first "partly according to the qualitative *sameness* of the moments that are in affinity, and partly according to the quantitative continuity" (emphasis added). The new measure thus has the same constituents as the first, with the same qualities in the same exclusive bond, though it unites different quanta of those constituents. It is a *new* measure, however, because the quanta of its constituents stand in a new ratio to one another – a ratio determined by those constituents to confer a new quality on their combination. Now since this new measure is itself a unity and relation between quanta, it is in turn subject to quantitative change, to which it is initially indifferent. There comes a point, however, at which such change turns it, too, into a new measure. Accordingly, "the new quality or the new something is subjected to the same progression of its alteration, and so on to infinity" (SL 320 / LS 412). The enduring exclusive unity of two independent measures thereby gives rise to a potentially infinite process of quantitative change, in the

course of which certain nodal points are reached at which their combination acquires a new quality and so constitutes a new measure, even though the qualities of its constituents remain the same.[18]

Insofar as the combination of measures undergoes a continuous change in quantity, different combinations are distinguished from one another "only by the more and less". To the initial quanta of the constituent measures a bit more is added or taken away, and then a bit more, and so on; in this respect, Hegel states, the change is "*gradual*" (*allmählich*). Such gradual change, however, simply proceeds from one quantum to another and does not *as such* entail any change in quality: "gradualness concerns merely the external side of the change, not its qualitative aspect" (SL 320 / LS 412). A new quality does not, therefore, arise in the nodal line purely because the quanta of the measures concerned *gradually* increase or decrease. It arises because such gradual change reaches the point at which a certain quantitative ratio is connected with a new quality by the *measures* in the combination. It is thus these measures, not gradualness itself, that explain the new quality. (The gradual change in quantity of A and B produces a new qualitative combination at a certain point only because it is the gradual change in quantity *of A and B* with their distinctive qualities; the same change in the quantities of C and D would not necessarily produce a new qualitative combination of them.)

Furthermore, the birth of the new quality actually breaks the chain of gradual change, for it introduces a qualitative *limit* into the latter: a point at which one measure stops and another begins. Change is gradual insofar as it involves continuously adding, or taking away, more of the same; qualitative change, by contrast, involves something *no longer* being the same but becoming something else (as it goes beyond its limit).[19] Such a change in quality in the nodal line thus interrupts gradual change with an immediate and sudden switch from this quality (and measure) to that. Hegel adds that the new something has its own quantum (or quantitative ratio) that is indifferent and external to that of the previous something. The change in quality thus takes the form of a "*jump*" or "*leap*" (*Sprung*) from one measure to another that is qualitatively distinct from the first and quantitatively indifferent to it (SL 320 / LS 412). This jump is preceded by a gradual change in the quantum of the first measure (that is, in the quanta of its constituent measures), but it cannot be explained by that gradual change alone and is not itself a gradual change.[20]

It is often said, Hegel remarks, that "there are no jumps in nature"[21] but that all change is simply a "*gradual* emerging and vanishing" (SL 322 / LS 414-15). The logic of measure shows, however, that *there must be such jumps*, since in certain cases gradual changes in quantity themselves lead to sudden changes of quality.[22] It is worth emphasizing here that, for Hegel, qualitative change and gradual quantitative change are not completely separate from one another, but the former is, indeed, a consequence of the latter. His claim, however, is that

the former is not simply *reducible* to the latter, since it interrupts it. In Hegel's view, therefore, change in the world is not always just quantitative, but changes in quantity also give rise to abrupt changes in quality at a point that is set by the distinctive measure of the thing concerned. It is this abrupt "transition" of quantity into quality that explains why there are "jumps" in nature. As Hegel puts it, "the changes in being as such are not only the transition of one magnitude into another magnitude, but the transition from the qualitative into the quantitative and vice versa, a becoming-other that is an interruption [*Abbrechen*] of the gradual and something qualitatively other than the preceding existence" (SL 322 / LS 415). So, for example, "water, in cooling, does not become hard little by little, such that it would become like porridge [*breiartig*] and then gradually harden to the consistency of ice, but it is hard all at once".[23]

Hegel's comments on the relation between gradual quantitative and sudden qualitative change echo what he says at the start of his account of measure. There is, however, a difference between the change we encountered earlier and the change we are now considering. In 1.3.1.A Hegel explains that gradually changing the quantum to which a given quality is attached can lead to the sudden demise of that quality and the thing it characterizes. Now, in 1.3.2.B, he explains that a *combination of measures*, that retain their exclusive affinity for one another, can also undergo gradual quantitative change that brings with it abrupt qualitative change.[24] This difference is in turn accompanied by another.

The bare thought that changing the quantum of something can destroy it is in itself just that bare thought. As we know from the logic of quality, however, the demise of one finite something yields a new something and so on *ad infinitum* (see SL 108 / LS 134). The thought that simple quantitative changes can destroy something thus *implicitly* contains the further thought of a nodal series of qualitative changes. By contrast, the thought of such a nodal series is *explicit* in the idea that qualitative changes are brought about by changes to the quanta of an enduring exclusive combination of measures. First, such a combination is inherently divisible and its constituent measures inherently changeable, and so it necessarily contains the possibility of *new* combinations of quanta of those measures. Second, the qualities of those measures endure and so continue to specify the quanta of further combinations. Third, as we have seen, that specification takes two forms: it consists in allowing quantitative changes to occur (and new combinations to arise) that preserve a certain ratio, *and* in determining that certain new quantitative ratios will confer new qualities on new combinations. Accordingly, a nodal series of measure-relations is contained explicitly in the idea that quantitative changes to an enduring exclusive combination of measures bring about changes in quality.

In the remark following 1.3.2.B Hegel then gives examples of such nodal series. These are provided by measures that combine with one another not

just in one fixed ratio – like the acids and bases in Fischer's table – but in several different ratios.[25] Oxygen and nitrogen, for example, when mixed in different proportions, form different compounds, such as nitric oxide, nitrous oxide and nitrogen dioxide, each of which has a different quality. Similarly, metals, such as lead, combine with oxygen to form various oxides with different qualities.

> *Metal oxides*, e.g. the lead oxides, are formed at certain quantitative points of oxidation and are distinguished by colors and other qualities. They do not pass into one another gradually; the proportions lying in between those nodal points yield nothing neutral, no specific existence [*Dasein*]. Without having passed through intervening stages, a specific compound appears that rests on a measure-relation [*Maßverhältnis*] and possesses its own qualities.
>
> —SL 321 / LS 414

In his lectures on the philosophy of nature Hegel gives a further example (taken from Berzelius) and states the actual proportions in which elements must be combined: "to saturate 100 parts of tin as protoxide, 13.6 parts of oxygen are required, as white deutoroxide, 20.4 parts, as yellow hyperoxide, 27.4 parts" (EN 263 / 326 [§ 333 A]).[26]

Note that in each example, different compounds arise because the *same* two substances combine in different ratios. Furthermore, they form nodal points on a continuous scale of quantities. In each case, the two elements can combine in greater or lesser amounts, but, provided that the relevant ratio between those amounts is preserved, the same compound is produced. If, however, the amounts combine in a different ratio that is the measure of a new kind of compound, then that new compound arises. The latter does not emerge gradually as the quantities are altered, but it is absent one moment, and present the next. A gradual change in quantity thus produces a sudden change in the quality of the compound.[27] (Note, by the way, that we do not need always to conceive of such change as a single process in which one compound is actually transformed in time into a different one, though the change may take that form.[28] What the logic of measure makes necessary is simply a "scale" [*Skala*] of different combinations of the same two elements, some of which differ only quantitatively from one another, and some of which differ qualitatively [SL 319 / LS 411].)

The examples above are not, of course, themselves discovered by pure logic. Logic shows that *measures* with a nodal structure are made necessary by the nature of measure itself and so must form part of the fabric of being, and it then falls to natural science to find suitable examples of such measures. Not all of Hegel's examples, however, exhibit the logical structure of the current nodal measure to the same degree, or as successfully, as others. He points out, for

instance, that changes in temperature cause water to pass through "the states of solid, liquid and vapor", and, as we saw above, that "these different states do not appear gradually" but arise through a "jump" (SL 321 / LS 414). Yet although the two constituent elements of water – hydrogen and oxygen – remain the same (as logic requires), its different states and their qualities are not distinguished by any change in the quantitative ratio between those elements. The changing states of water do not, therefore, correspond exactly to the logical structure of the current measure.

Hegel also claims that "the system of natural numbers already exhibits such a *nodal line* of qualitative moments which emerge in a merely external progression" (SL 320 / LS 413). This claim is surprising because natural numbers are simply fully determinate quanta, and Hegel has insisted that the nodal line characterizing the current measure is precisely *not* just a sequence of quanta. He points out, however, that, besides forming a simple arithmetic sequence, natural numbers also have a "*specific* relation" to others in the sequence, that is to say, a relation that is qualitatively different from just being *bigger* or *smaller*. Such a relation is enjoyed by numbers that are multiples or, more significantly, *powers* or *roots* of other numbers. A power and its root are numbers like any others, but they have a qualitative, as well as a quantitative, connection, since the root relates to *itself* in its power and thereby exhibits the quality of being-for-itself (see SL 278-9 / LS 359-60). Such qualitative connections between powers and roots arise at certain intervals in the sequence of natural numbers and in so doing *interrupt* what is otherwise a merely arithmetic, quantitative sequence. Accordingly, they form nodes in that sequence, and in that sense natural numbers, which in themselves are nothing but quanta, can be said to exhibit the structure of the current measure.

THE MEASURELESS

The exclusive measure, as it is now conceived, is not just an independent combination of two quantitative ratios, such as densities. Yet nor is it just a unity of two measures – such as an acid and base – that have a qualitative, "elective" affinity for one another and combine, or saturate one another, in a single quantitative ratio. It is a qualitative unity of two measures – such as oxygen and nitrogen – that combine in a specific ratio, but that can also combine in a series of further, different ratios and so form a "nodal line".[29]

Each combination of such measures has its own distinctive quality or "being-for-self" (SL 323 / LS 416): it is, for example, nitric oxide, nitrous oxide or nitrogen dioxide. As just indicated, however, each combination is distinguished by a specific quantitative *ratio* between its constituent elements. Since each is thus produced by uniting different *amounts* of the two elements concerned, each is, as Hegel puts it, "tainted by the moment of quantitative existence

[*Dasein*]". Amounts or quanta, however, are inherently variable. The amount of each element in their combination can thus always be changed, or, in Hegel's words, the latter is necessarily "capable of movement up and down the scale of the quantum". As we have seen, such change in turn can have the following outcomes: it can preserve the quantitative ratio between the two amounts or it can change the ratio into one that characterizes a new measure.

A third alternative is that a new ratio arises that does not characterize a new measure, in which case the original measure will remain and any excess of one element will just be an unabsorbed residue. If, however, the new ratio that arises does belong to a new measure, then the original measure will be "destroyed by the mere alteration of its magnitude"; that is, it will be transformed by that alteration into its *negation*, and so into what is from its point of view *without* measure. As Hegel puts it, therefore, it will be "driven beyond itself into the *measureless* [*das* Maßlose]" (SL 323 / LS 416). Note that this is a subtly different case from the one we encountered at the start of the logic of measure. There we saw that an entity can be destroyed by simply becoming too big or too small, but in the case we are now considering, a compound measure is destroyed when the *ratio* that constitutes it is displaced by another.

The loss of this ratio, however, leaves the elements in the compound in a new ratio that is the measure of a new compound of them. The "measureless" relation between elements that lies beyond the measure they form thus constitutes another *measure*. In Hegel's words, "the new measure-relation into which the original passes over is measureless with respect to the latter but, in its own self, it is equally a quality that exists for itself [*für sich-seiende Qualität*]" – that is, a compound with a new quality (SL 323 / LS 416-17).[30] This new measure is, however, also susceptible to further quantitative changes, some of which will lead to the emergence of new measures, and these will themselves be susceptible to further changes. In this way, Hegel writes, "there is posited the alternation of specific existences [*Existenzen*]" – of measures – "with one another and of these equally with relations that remain merely quantitative", and a nodal line of measures is generated by the elements concerned.

This nodal line, however, is now – in 1.3.2.C – understood in a slightly different way from the way it was conceived in 1.3.2.B: for each new measure in the line is now at the same time an explicitly measureless "beyond" from the point of view of the one that precedes it on the scale. Moreover, since quanta can always be changed and the nodal line is thus in principle endless, the latter is itself without limit or "measure" (though it may not always be endless or "measureless" in nature).[31] As such, of course, the nodal line constitutes the bad infinite, or infinite progress, in the sphere of measure: a series of measures that extends indefinitely into a measureless beyond.[32] Yet Hegel argues that this bad infinity also contains true infinity, or "the *infinite* that is for itself" (SL 323 / LS 417). Each measure in the nodal series is itself independent, exclusive and so

"for itself". It turns out, however, that the endless *series* they form also exhibits the quality of being-for-self and thus of true infinity. Hegel clarifies the nature of true infinity in the sphere of measure by comparing it briefly with both qualitative and quantitative infinity.

In the sphere of simple quality the finite and the infinite turn logically *into* one another; each is thus the "*transition*" (*Übergang*) of itself into its opposite (SL 323 / LS 417). The finite first proves to be infinite because, in coming to a definitive end, it gives rise to another finite being, and another, and so in fact constitutes continuous, *unending* being. This infinity, however, is the opposite, and the negation, of the finite and so (when this negation is rendered explicit) places itself beyond the latter. In so doing, of course, it turns itself into a limited, *finite* infinite. As finite, however, it points beyond itself to the infinite that is *its* negation. This infinite is then once again finite and so points beyond itself, and so on. The finite and infinite thus give rise to an "infinite progress", in which each points beyond itself to, or proves to be, its immediate opposite. Yet this progress or movement is not simply one of tedious endlessness, for it also contains what Hegel calls "true" infinity. In that progress, each of the two participants not only points beyond itself to, or turns into, its opposite, but in so doing it relates to and unites with *itself*: the finite points to an infinite beyond that is itself finite, and the infinite, in its finitude and limitation, points to a further infinite. As Hegel puts it, "both the finite and the infinite are thus this *movement* of each returning to itself through its negation" (SL 117 / LS 147). This process of relating to oneself in and through one's negation is true infinity. The finite infinite is thus not itself the true infinite, but, together with the finite as such, is merely a moment of the latter.

The detailed logical structure of qualitative infinity is set out in volume 1, chapter 11. What needs to be noted here is that such infinity is the process of relating to oneself in passing over into one's *opposite*. There is an immediate qualitative difference between the finite and its negation, the infinite; yet in pointing to, or proving to be, its opposite, each in fact relates to itself and thereby constitutes truly infinite being. Quantitative infinity, by contrast, has a subtly different logical structure. It, too, entails relating to oneself in another, though the "self" involved here is a quantum, rather than a mere finite (or finitely infinite) something. The other quantum, however, is not merely the negation or opposite of the first, but is the explicit continuation of the latter: it is the first quantum *again* (albeit in a different form). In Hegel's words, quantitative infinity is "the *continuity* of the quantum, a continuity of it beyond itself" (SL 323 / LS 417). Such infinity is thus manifest above all in the raising of a quantum to a power of itself: for the power of a quantum is simply a further form *of* that quantum, produced *by* the quantum itself.

True infinity in the sphere of measure is to be found in the nodal line that we have been describing. Since such infinity is proper to *measure*, its moments are

neither merely qualitative nor merely quantitative: they are not just finite somethings, nor pure quanta. Nonetheless, such infinity exhibits features of both qualitative and quantitative infinity. First, this infinity incorporates the logical *transition* – integral to qualitative infinity – of two opposed moments into one another. Here, however, these two moments are the two constituents of measure as such, namely quality and quantity themselves. They pass into one another in the nodal line in the following way: the qualitative change from one measure to another proves *not* just to be qualitative after all, but to be brought about by the increase or decrease in quanta; and yet this quantitative change itself proves *not* just to be quantitative, precisely because it leads (or can lead) to the emergence of a new measure with a new quality.[33] In the nodal line, therefore, quality and quantity turn into one another by being *negated*; or, as Hegel puts it, "this infinity of the specification of measure *posits* both the qualitative and quantitative as *sublating* themselves into one another" (SL 323 / LS 417). (In this way, quality and quantity *prove* to be a unity in the nodal line, rather than just forming an immediate unity, as at the start of the logic of measure.)

Yet although this transition of quality and quantity into one another is integral to true infinity in the sphere of measure, it does not exhaust such infinity. True infinity also requires, indeed is properly constituted by, the second feature of the nodal line: the moment of *continuity* that we saw in quantitative infinity. Here, however, what continues is not an "infinite", self-relating quantum, but the *quality* of the two elements whose several combinations form the nodal line.[34] This line consists of different measures which each have a different quality or character; yet they are all combinations of the same two elements; in the change from one measure – and so one quality – to another, therefore, the qualities of the elements remain unchanged. Since they continue without change, and *without end*, throughout the whole course of the nodal line, they provide the explicit moment of true infinity in the latter.

It is important to stress that the *continuity* of quality exhibited in the nodal line is inseparable from the *transition* of quality and quantity into one another, and does not constitute a "bad infinite" beyond the latter. This continuity is tied especially to the logical transition of quality into quantity. In the nodal line, as we know, the quality of one measure gives way to that of another measure, and so there is a change in quality. Yet that change in quality is brought about by a change in *quantity* (and, more specifically, in ratio): quality thus passes over logically into its immediate opposite. Changes in quantity, however, are ones to which quality is *indifferent* and which thus allow quality to remain what it is. In the quantitatively generated change of quality from one measure to another, therefore, there must be quality that remains *unaffected* by the change and that continues simply being what it is. This continuing quality is that of the elements that enter into different combinations.

We saw earlier (2: 318-19) that these elements remain the same because they form an exclusive unity that is variable *within itself*. This unity can thus be transformed into (or broken apart and reconstituted as) a series of different unities without ceasing to be the unity that it is. Now we learn that the elements remain the same for a further reason: because the change from one qualitative unity to another is in fact a change in quantity to which the elements are indifferent. In the change from one qualitative unity to another, therefore, the quality of the elements simply continues in a new form and so relates only *to itself*. Hegel puts the point like this:

> The transition of the qualitative, of a specific existence [*Existenz*], into another, is such that what happens is only an alteration of the quantitative determinacy of a ratio;[35] the alteration of the qualitative itself into what is qualitative is thus posited as an external and indifferent alteration and as a *going together with itself* [*Zusammengehen mit sich selbst*].
>
> —SL 323-4 / LS 417

As indicated above, it is in this "going-together-with-itself", or *continuity*, of qualitative being that explicit *true infinity* is to be found in the nodal line. True infinity resides in the fact that the elements concerned continue to form a unity even though that unity itself takes different forms. Note again that such infinity combines features of both qualitative and quantitative infinity (as befits the sphere of measure, in which quality and quantity are united). True qualitative infinity is exhibited when one moment relates to, and unites with, itself in pointing to or proving to be its qualitative opposite. True quantitative infinity is exhibited when one quantum continues explicitly to be itself in the other quantum that it becomes (for example, by raising itself to a power of itself). True infinity in the sphere of measure is exhibited when *quality continues* uninterrupted as it takes on forms that are quantitatively, but thereby also qualitatively, different from one another.[36]

Hegel goes on to note that the qualitative unity of elements that continues throughout the change of measures is "the truly persisting, independent *matter* [*Materie*] or *thing* [*Sache*]" (SL 324 / LS 417).[37] It is the self-relating being that remains what it is *for itself* as one measure displaces another. As such, Hegel writes, it constitutes the "perennial substrate [*Substrat*]" of change (SL 324 / LS 418). This is the same substrate or "qualitative foundation" we encountered in 1.3.2.B, only now identified with the moment of *true infinity* in the (potentially) endless and measureless nodal line of measures (see SL 319 / LS 411). Why, though, should this persisting matter be conceived as the "substrate" or "foundation" (*Grundlage*) of the nodal line, that is, as something *underlying*, and so distinct from, the process of change itself? This is a question we did not consider in the previous section, but that now needs to be addressed.

The question needs to be considered, because the persisting matter in the nodal line should surely be conceived as *co-extensive* with, rather than the distinct substrate of, the process of change. Things are, however, not quite so simple: for those qualities persist as their *quanta* change, and such quantitative change is that to which the qualities are explicitly *indifferent*. It is true that the qualities of the new measures that emerge are not indifferent to such change, since they are generated by the latter; but the case is different with the qualities that persist and form those new measures. As Hegel puts it, they are "the continuing of the qualitative into the quantitative progression as into an *indifferent* [*gleichgültig*] alteration" (SL 324 / LS 418, emphasis added). It is this moment of "indifference", I think, that renders quantitative change external to the persisting qualitative matter and in so doing turns the latter into something immediately distinct from the former, that is, into a "substrate". In this sense, the matter that persists in the nodal line has something of the bad infinite or the beyond about it after all.

In the sphere of quality something preserves its identity in the course of its own change, but that identity does not "underlie" the latter (see SL 92 / LS 114). Similarly, the quantum does not "underlie" the process in which it changes into further quanta, but its continuing beyond itself in another quantum *is* its very process of change: it is the increasing or decreasing of *itself* (SL 189 / LS 240). In the nodal line, by contrast, a *qualitative* matter persists as its *quantum* changes, and since it is indifferent to the latter and the latter is in that sense external to it, it takes the form of a substrate underlying such change. The difference between quality and quantity, therefore, is what turns persistent matter into a substrate. Hegel notes that such a "detaching of being from its determinacy" actually begins "in the quantum as such", since something with a magnitude is "indifferent" to that magnitude (SL 324 / LS 417).[38] In the logic of quantity, however, the topic is the development of quantity as such, so the character of the qualitative "something" that has magnitude is not considered any further (see 2: 153). Now, by contrast, the subject of logic is the explicit unity of quantity and quality; and, moreover, quality is understood specifically as that which persists through quantitative change. Since such quality is also indifferent to that change, it necessarily takes the form of a substrate.[39] This is true in both 1.3.2.B and C.

Note, however, that this qualitative substrate is not yet a form of *essence*, which is the subject of the second book of the *Logic*. This is because it does not reduce quantitative change to merely illusory being or produce it via a process of mediation.[40] The substrate still belongs to the sphere of *being*, since it is just immediately different from (insofar as it is indifferent to) such change. As Hegel puts it, "the perennial substrate" has initially "the determinacy of infinity *that is* [*die Bestimmung* seiender *Unendlichkeit*]" (SL 324 / LS 418, emphasis added).

It is worth noting, by the way, that there is an element of continuity in the very fact that the nodal line consists in a series of changing quanta and quantitative ratios: for quantity is itself the continuity of discrete units (see SL 154-5 / LS 194-5). The enduring substrate to which Hegel draws our attention here is not, however, reducible to this quantitative continuity. It consists in the *quality* that remains indifferent to quantitative change and for this reason persists without alteration throughout the nodal qualitative changes that quantitative change brings about. To quote once again words from Hegel quoted above (only this time with a different emphasis), the substrate is "the continuing of the *qualitative* into the quantitative progression as into an indifferent alteration" (SL 324 / LS 418).[41]

Yet quality not only proves to be continuous in the nodal line, but, as we indicated above, it is also subject – with quantity – to *negation*. As Hegel explains, quality and quantity are negated in the nodal line because each proves, logically, not just to be itself but to be its other: quantitative change leads to a new measure with a new quality, but that quality itself rests on a new quantitative ratio. Or, in Hegel's own words, "the quantitative pointing beyond itself to another which is itself quantitative perishes [*geht unter*] with the emergence of a measure-relation, of a quality, and the qualitative transition is sublated in the very fact that the new quality is itself only a quantitative relation" (SL 324 / LS 418).[42] We noted above that the continuity of quality in the nodal line is inseparable especially from the logical transition of quality into quantity. As we also indicated, however, that continuity – and the infinite being that it constitutes – is in fact inseparable from the transition of quality and quantity *into one another* and from the consequent negation of each (2: 331).

The quality that is negated is, of course, not the same as the one that continues. The latter is the quality of the two elements that form different measures; the former, by contrast, is the quality of each individual measure that is produced by a new ratio between those elements. Note, too, that the *negation* of quality and quantity, and their logical transition into one another, coincides with the *progression* along the nodal line from one measure to another: for it is precisely in the production of a new measure that quantity passes over logically into quality and quality into quantity. The continuous quality of the elements is thus itself inseparable from the progression from measure to measure – from the *change* in the quality of the measures and the *change* in quantity that brings about the latter. Indeed, as Hegel points out, it is the very process of change that shows the quality of the elements to be a *continuing* substrate. The elements endure because they form a unity that can vary without ceasing to be; the only way they can *prove* to be enduring, however, is by remaining *as* different forms of their unity are produced and then surpassed. The process of change – and the transition of quality and quantity into one another with which that process coincides – is thus *itself* "the *showing* or *positing* that such a substrate underlies it" (SL 324 / LS 418).

Moreover, change and continuity are even more intimately connected in the nodal line, since the changing measures are simply different combinations *of* the enduring elements. The measures are surpassed and so "negated", whereas the elements remain in being, but the former are not for that reason something separate from the latter. Rather, those measures are themselves different "states" (*Zustände*) of the substrate formed by the elements. In the nodal line, therefore, "*that which undergoes transition* [*das Übergehende*] is posited as remaining the *same* in the process", while "the measures and the independent entities thereby posited are reduced to *states*" of the latter (and "alteration is only a change of state") (SL 325 / LS 418).

The substrate thus in fact stands in an ambiguous relation to the process of change. On the one hand, it is indifferent to, and so underlies, the process of quantitative change (and thereby resembles the bad infinite). On the other hand, such quantitative change yields new measures with new qualities and these are themselves new forms *of* the persisting qualities. The substrate does not, therefore, just "underlie" change, but in another respect it also coincides with such change insofar as it differentiates itself *into* the series of changing measures. As Hegel puts it,

> This sameness of the substrate is *posited* in the fact that the qualitative independent measures into which the measure-determining [*maßbestimmend*] unity is dispersed consist only of quantitative differences, so that the substrate continues into this differentiation of itself [*dieses sein Unterscheiden*].
>
> —SL 324 / LS 418

In this latter respect, the "substrate" can be said to be the process of its own moments, like true infinity in the sphere of quality.[43]

Note, by the way, that the substrate is conceived at one point in 1.3.2.C as the unity of quality *and* quantity, even though, as we have seen, it is also said to be a "qualitative foundation" and to consist in the "continuing of the qualitative" (SL 319, 324 / LS 411, 418).[44] There is, however, no contradiction here. In the nodal line *quality* is indeed the continuing moment and forms the substrate since it is indifferent to quantitative change; yet it is itself always present in some quantity, and it continues or "goes together with itself" only in and through quantitative change, so the substrate is in fact also the unity of quality and quantity (even if no specific quantitative ratio between its components endures).[45]

Hegel also maintains that with the emergence of true infinity in the sphere of measure – in the form of an ambiguous substrate – the "first immediate *unity*" of quality and quantity, which measure as such is, is posited as having "returned into itself" (SL 323 / LS 417). What this means, I take it, is that the unity displayed by measure at the start (in 1.3.1.A) has now been restored,

albeit in a more complex and internally differentiated form. This idea is helpful in allowing us to see the overall pattern of measure's logical development.

At the outset, measure is a unity in two senses: it is the unity of quality and quantity, but in the form of a quantum (or range of quanta) that *stands alone* and to which a given quality is attached. The unity of measure is then lost, or at least partly undermined, as the latter proves to be a *relation* or *ratio* (*Verhältnis*) between two entities – entities that are distinguished first as measure and quantum (1.3.1.B.b) and then as two "specified" quanta or powers (1.3.1.B.c). Measure (as real measure) then proves to be the "combination" (*Verbindung*) of two ratios and at this point becomes once again an explicit unity (SL 304 / LS 391).[46] Each ratio in the combination, however, has an exponent that is a simple quantum; a quantum in turn is not just something in relation to another, but is a *one* that is one of many; each ratio thus forms a *series* of unities with several other ratios and the overall unity of measure is again lost (1.3.2.A.b). With the emergence of the nodal line and its *continuous substrate*, however, the unity of measure is again restored – albeit as a unity of changing moments or "states" that are themselves measures. The task now is to render explicit what is implicit in the idea of such a substrate.[47]

CHAPTER FOURTEEN

Indifference

ABSOLUTE INDIFFERENCE

With a nod toward Schelling, Hegel names the substrate that has emerged at this point in the *Logic* "indifference" (*Indifferenz*) (SL 326 / LS 420).[1] Pure being and pure quantity can also be described as indifference, since the former lacks all difference and determination, and the latter is "capable of all determinations" but indifferent to those that may arise: quantity does not have to take the form of any particular quantum. The substrate that has now emerged, however, differs from both of these and is called "absolute" indifference. It does not just lack determinacy, nor is it simply open to different determinacies, but it *endures* through a series of changing determinations in explicit indifference to them.

As the "continuing of the qualitative", this substrate is indifferent to quantitative changes (SL 324 / LS 418); but it is also indifferent in a broader sense, since it endures as all three principal determinations of being – quality, quantity and measure – change and thereby suffer negation. Note, however, that it does not subsist at one remove from the determinations of being, but is inseparably bound to them: for it proves to be an indifferent substrate only by remaining unaffected *as* quality and quantity turn logically into one another and one measure is thereby replaced by another. Absolute indifference is thus not a wholly free-standing immediacy, but, as Hegel writes, it "*mediates itself with itself*" "*through the negation* of all determinacies of being, of quality and quantity and their at first immediate unity, measure" (SL 326 / LS 420).

The indifferent substrate is bound to such determinations in a further respect, since the different measures that arise in the nodal line, with their different ratios and qualities, are states *of* that substrate. These states, however, are also described by Hegel as "*external*" to the substrate due to the fact that the latter

is *indifferent* to, and so unaffected by, those states. Thus, even though the states belong to the substrate, it also differs from them and has an identity of its own that remains untouched by them. Indeed, this *difference* between the substrate and its states is precisely what makes it their *sub-strate*. Whereas the states themselves change, the substrate does not change but endures; it is thus indifferent to, and so immediately different from, those states, and, as such, underlies them. In Hegel's words, "the determinacy is in it still only as a state, that is, as *something qualitatively external* that has indifference as its *substrate*" (SL 326 / LS 420).

As we noted in the last chapter, therefore, the indifferent substrate of the nodal line is logically ambiguous: it underlies determinations that are also states *of* it (2: 335). Hegel now argues, however, that the difference between the substrate and its changing states is in truth even less sharp than has been suggested so far. This occurs in the second paragraph of 1.3.3.A.

Hegel first reminds us that the qualitative "as thus external to being" – as a *state* of the continuing substrate – "is the opposite of itself and, as such, is only that which sublates itself" (SL 326 / LS 420). Such quality is the "opposite of itself" in the nodal line, because it is generated by *quantitative* change: the quality of a new measure in that line emerges simply because the elements concerned are combined in a new quantitative ratio. Note that this quality is not an illusion, but is, indeed, produced by quantitative change. This means, however, that quantity itself immediately turns into *its* opposite: quality. Each, therefore, turns into the other, so the clear difference between them is undermined. As Hegel puts it, that difference proves to be "an empty differentiation" (*ein leeres Unterscheiden*): quality and quantity differ, but in such a way that their difference is also negated (SL 326 / LS 420).

The undermining of the clear difference between them does not, however, leave us with nothing: for there is being that continues regardless and is thus *indifferent* to that self-negating difference. Such being, as we have seen, is the "continuing of the qualitative". This continuing being forms what has been understood so far to be the enduring substrate of change in the nodal line. Yet closer attention reveals that it cannot simply be an underlying *substrate*, because it is what continues *in* the transition of quality and quantity into one another, rather than "beneath" it.

Quality and quantity in the nodal line are not like atoms separated by a void, but together they form a continuing line of being. Indeed, together they constitute the "continuing of the qualitative". The two original qualities that form the different nodes are what *continue*, but they do so only insofar as quality and quantity pass into one another and produce those nodes. Continuity is to be found, therefore, only *in* the one's becoming the other. Note that this thought does not just render the indifferent substrate of the nodal line ambiguous. It undermines the very idea of a "substrate" by fusing continuity

with the process in which quality and quantity undermine their difference from one another. Continuous, *indifferent* being – or "absolute" indifference – is thus not a substrate after all, but it is constituted by quality and quantity themselves *as* they undo the difference between them and thereby give rise to new measures. To put the point another way, the undoing of that difference is itself what causes the "qualitative" to *continue* and *remain* (rather than just be what it is).

This, I would argue, is what Hegel has in mind when he states that "precisely this empty differentiation is indifference *itself* as result" (SL 326 / LS 420, emphasis added). Indifference is external to its various changing states – to the changing qualities, quanta and measures in the nodal line – insofar as it is, indeed, *indifferent* to them, and those states are in turn external to it. Yet indifference is, equally, the continuous, indifferent being that is constituted as such *by* its changing states (since it is only in their *change* that it can *continue* "indifferently"). As Hegel puts it, "it is precisely externality and its vanishing which make the unity of being into indifference".

It turns out, therefore, that there are not *two* distinct, but related, dimensions to the nodal line – that of the states and that of their continuing substrate – but there is in fact just one, because the indifferent continuity itself persists *through* and is constituted *by* the changing states. Unity in the sphere of measure is thus even deeper than we thought, for not only is there explicit continuity in the nodal line, but such continuity is one with the very process of change that it appears to "underlie". This continuity is still indifferent to such change, but it runs through that change itself and so is not just the substrate of the latter. The changing, negated states in turn are now to be understood, not merely as distinct-but-also-inseparable-from their "substrate", but as actually falling *within* the continuity of indifference that they themselves constitute. Indifference is thus revealed to be "concrete", rather than abstract, for it incorporates within itself "the negation of all determinations of being". As Hegel puts it, indifference "contains negation and relation, and what was called a 'state' is its immanent, self-related differentiation"; the "external" states and their "vanishing" are therefore "*within* [*innerhalb*] this indifference, which thereby ceases to be only a substrate and *in its own self* only abstract" (SL 326 / LS 420).[2]

And yet, having said all of this, this newly disclosed identity of indifference and its states does not altogether remove the difference between them. This is because indifference is continuous, *rather than* changing being and is, precisely, *indifferent* and so external to its changing states. It thus retains the character of a "substrate" after all. This does not mean that no logical progress has been made in the course of 1.3.3.A. The logical development in that section should, however, be understood as follows. In the first paragraph of 1.3.3.A, the continuity of being in the nodal line is conceived as "mediated" by and so inseparable from the changing determinations of quality, quantity and measure, and the latter are understood to be states *of* that continuing being. Nonetheless,

that continuing being is essentially indifferent and external to those states and is, therefore, their underlying *substrate*.[3] This substrate and its states thus form *two* different levels of being. In the second paragraph of 1.3.3.A, continuing being is still indifferent and external to its changing states, and so in this sense remains their "substrate"; yet it is also at the same time *one* with them and contains them *within* its "concrete" continuity. The result that emerges in 1.3.3.A is thus the following: indifference and its states are no longer essentially *different* though inseparable, but they are, rather, two aspects of *one and the same* continuing being.[4] This result, however, not only discloses more about indifference in the nodal line, but it also makes visible the structure of a new kind of indifference altogether. This new indifference arises when we render explicit what is implicit in the initial indifference.

INDIFFERENCE AS THE INVERSE RATIO OF ITS FACTORS

The new indifference thus has the concrete structure that we have just discerned in the nodal line. It has this structure *explicitly* from the start, however, and for this reason it no longer permeates a nodal line of measures, but expresses itself in the "inverse ratio of its factors" (SL 327 / LS 420).[5]

The new indifference – due to being derived from the previous one – is, first of all, *one* with the changing determinations to which it is indifferent. As Hegel puts it, it is "their continuing into one another and hence the indivisible independent being [*das untrennbare Selbständige*] which is *wholly* present in its differences" (SL 327 / LS 420).[6] Since this indifference proves to be such *in* a process of qualitative and quantitative change, it is mediated by the latter. Yet it is precisely *indifferent* to such change and so remains external to it, and as such it is still the *substrate* of change. Accordingly, that mediating change is not contained explicitly in this indifference itself (even though the latter is one with the former). In Hegel's words, the substrate is "*implicitly* [*an sich*] mediation", but this mediation "is not yet posited as such *in it* [*an ihm*]" (SL 327 / LS 421). As we now see, it is this fact that indifference is both identical to *and* different from its mediating changes (and differences) that determines the latter to stand in an inverse ratio to one another.

Since the continuity of being is here indifferent to and unaffected by the different, changing moments, it is, as we have noted, *external* to the latter, and they are in turn external to it. External determinations, however, are first and foremost *quantitative* ones. The different moments to which indifference is indifferent must, therefore, be quantitative first of all. Such quantitative difference, however, is itself initially simple and immediate: the simple difference between this quantum and that one. As Hegel writes, therefore, "there are *two* different quanta of one and the same substrate" (SL 327 / LS 421, emphasis added). Yet, according to the logical structure we are now

considering, these two quanta are not just external to, and carried by, their continuous substrate, but together they constitute that substrate and so they are *one* with it. The substrate, therefore, must consist in the togetherness, or "sum" (*Summe*), *of* the quanta themselves and so be a quantum, too.[7] Hegel points out that the substrate, as *indifferent* to quantity, is not in itself a quantum: it is simply continuous, uninterrupted being with no particular identifying amount.[8] Yet insofar as it is the continuity or unity that is constituted *by* two simple quanta, it must be the *sum* of those quanta. This is not to say that the substrate is absolutely restricted to the amount of this sum – different quanta could yield a different sum – but *these* quanta yield *this* sum.[9]

It is important to stress that this sum is not just a third changeable quantum besides the first two but, as their *sum*, it is their continuing substrate. As such it reflects the logical complexity that characterizes the new indifferent substrate we are examining. On the one hand, that sum is constituted by, and so is one with, the two quanta; indeed, it *is* just the two quanta taken together. On the other hand, however, that sum also differs from those quanta and has an identity of its own: for it remains an *unaltered* amount as they undergo change. The two quanta, as such, are inherently variable and so can be replaced by other quanta; the sum that forms their substrate, however, continues to be the same throughout such change. So as 7 and 5 are replaced by 8 and 4, and then by 9 and 3, the sum is always 12, and the latter is thus the substrate that remains indifferent to the quanta that make it up. As such a substrate, Hegel notes, this sum sets an "absolute limit" to the change that the quanta can undergo, and in this way constitutes the governing *measure* of such change.

Thanks to the fact that the sum differs from its constituent quanta, it stands in *relation* to them and, as just noted, limits their change. At the same time, it stands in relation only to the quanta that constitute it: it is only *their* limiting substrate. Hegel pulls these thoughts together by stating that "indifference is this fixed measure, the intrinsic [*ansichseiend*] absolute limit, only in *relation* to those differences" (of which it is the sum) (SL 327 / LS 421). He then goes on to argue that this invariant and indifferent limit requires the two quanta to stand in an *inverse ratio* to one another: for if one quantum increases, while the sum remains the same, then the other must decrease by a proportional amount.[10] Note, however, that this inverse ratio differs subtly from the one we encountered in the section on quantity. It does so, because it is not governed by a numerical exponent that is explicitly and *solely* a function or product *of* the two quanta concerned; rather, it has a "real substrate" in the simple sum of the quanta – a sum that is itself merely a given amount (SL 327 / LS 421). A limit is thus set to the change of the two quanta by this amount or *sum* (a + b), rather than by their *product* (a x b). Yet the effect of that limit remains the same: as one quantum increases, the other must decrease. The two quanta thus negate (or are negated by) one another as they undergo change.[11]

Note, though, that the quanta negate one another *directly* in a way that does not occur in the earlier inverse ratio. In that earlier ratio, when one quantum increases it does not reduce the other by the amount of its own increase; rather, it reduces the other by the amount required to preserve the product of the two. So, if 2 x 12 = 24, and 2 is then increased by 1 to 3, 12 is not similarly reduced by 1 to 11, but it is reduced by 4 to 8, since 3 x 8 = 24. In the current ratio, by contrast, an increase in one quantum decreases the other *by the same amount*, since it is their sum, not their product, that is to be preserved. In this sense, each quantum is the direct negation of the other, and does not just negate the other indirectly via their product: each sets its *own* limit to the other through its increase (or decrease). This means, Hegel explains, that the two quanta actually stand in a *qualitative* relation to one another. In the sphere of quality one something stops where another begins; the being of one is thus the *non*-being of the other. The same is true of the two quanta in the current inverse ratio: one directly negates the other and so in that sense is qualitatively different from it. “Restricted by the fixed limit of their sum”, Hegel writes, they relate “negatively to one another, and this is now the qualitative determination [*qualitative Bestimmung*] in which they stand to one another” (SL 327 / LS 421). This qualitative difference between the quanta cannot, however, consist in their being different *kinds of quantum* – for example, a simple quantum and a power – for they differ qualitatively precisely in both being *simple* quanta that negate one another through the amount by which they increase or decrease. The two quanta must, therefore, differ qualitatively by bringing with them two distinct and simple *qualities*. Thus, as befits the sphere of measure, the difference between the two quanta is at the same time a qualitative difference, a difference between quantitative moments with different associated qualities.[12]

This is not to deny that there can be inverse ratios between quanta, governed by their sum, in which abstraction is made from all quality – ratios that are purely quantitative. Here, however, we are concerned with logical necessity, not just with what is possible. As we have seen, the logic of indifference makes necessary an inverse ratio between quanta in which they are limited by their sum and so directly negate one another. We have now seen that this feature of the ratio in turn requires the two quanta to stand in a qualitative, as well as a quantitative relation to one another. So, although purely quantitative inverse ratios are of course possible (if we abstract from quality), the inverse ratio that belongs to indifference must coincide with a relation between two distinct qualities.

These qualities, however, do not have a separate, independent existence, but are inseparable from one another in their difference. They arise, logically, because the two quanta in the ratio, in changing, negate and limit *one another*, so they too must be bound to one another. The qualities also form a unity for

another reason: because the quanta to which they are attached form a *sum*. This in turn has a further significant consequence: for, since the unity of the qualities coincides with the sum of the quanta, it must constitute, with the latter, the continuing indifferent *substrate* of the different moments. There is, however, an important difference between the substrate conceived *as* a qualitative unity and conceived *as* a sum.

The substrate as such is that which remains indifferent to the different moments with which it is associated, but it is also one with those moments. The sum of the two quanta embodies this structure because it remains constant as they change and it is, precisely, the *sum of* the quanta: it is the two *as* one. Note, however, that it is not, and cannot be, present *as* that sum in either quantum taken by itself, for both are required for their sum. The substrate as qualitative is also the unity of the two moments involved, but it is not just the sum *of* them. It is the qualitative unity that is *indifferent* to the quantitative difference between the moments and so continues unbroken *in* both of them. As such, in Hegel's words, it is "the indivisible independent being which is *wholly* present in its differences" (SL 327 / LS 420). Unlike its quantitative counterpart, therefore, the substrate as a qualitative unity is, and must be, present in each of its mediating moments.

The indifferent substrate thus takes two distinct forms: it is the sum *of* its associated moments (insofar as they are quanta), as well as the "self-sameness of the qualities" *in* each moment (SL 328 / LS 422). This produces the following complex logical structure. There are two quanta with one and the same substrate, but each comes with its own distinctive quality and, as such, forms one "side" of the relation between them: one side, X, thus has the quality A, and the other side, Y, has the quality B. At the same time, the unity of A and B constitutes the continuous indifferent substrate that is present *on both sides*. Both X and Y, therefore, must in fact contain both qualities: "the substrate itself, as indifference, is in itself likewise the unity of the two qualities; each of the sides of the relation, therefore, equally contains them both within itself" (SL 327 / LS 421). What differentiates the two sides cannot, therefore, be their distinctive qualities alone, but must be the *quanta* of the latter they contain. That is to say, one side can be "distinguished from the other side only by a more of the one quality and a less of the other, and *vice versa*". So X must differ from Y by containing more of A than B, whereas Y contains more of B than A. The two sides do not, therefore, contain only their own qualities, but "the one quality is, through its quantum, only *preponderant* [*überwiegend*] in the one side, as is the other quality in the other side".[13]

Note that this complex structure is made necessary by the nature of *indifference*, as it is here understood. Such indifference, as we have seen, requires there to be a difference between two quanta with different associated qualities; yet it is also the unity of these qualities that is indifferently present on

both sides. The sides can only be distinguished, therefore, by one of them having more of one quality than the other. In this respect, as just noted, the substrate as a qualitative unity itself differs logically from the substrate as a quantitative sum: for the sum of two quanta cannot be explicitly present in each of its two constituents. If it were, then each constituent would itself disappear to be replaced by their sum, and the sum of the two of them would thus actually be their sum twice over. Each quantum can be considered to be *implicitly* the sum of the two, insofar as it is bound to its counterpart in an inverse ratio, but it cannot be that sum explicitly (see SL 327 [ll. 29-30] / LS 421 [ll. 27-8]).

Hegel goes on to note that "each side [of the inverse ratio] is thus in itself an inverse ratio" between the quanta of the two qualities it contains (SL 328 / LS 422). This is because, in any *given* ratio between X and Y, the quanta of A and B in either side can change but in so doing must preserve the amount that defines the side, so that the ratio between the sides is itself preserved. As the amount of A in X increases, therefore, the amount of B in X must decrease accordingly, so that X itself remains the same quantum in relation to Y. Such changes must, of course, also preserve the "preponderance" of one quality over the other that gives a side its distinctive overall character and quality (or "quality").

To give an example, take two sides, X and Y, that together form a sum of 30 units, and let X = 20 and Y = 10. Then let the 20 units in X comprise 12 units of A and 8 of B, whereas the 10 units of Y comprise 4 units of A and 6 of B:

X = 20	Y = 10
(**A = 12**; B = 8)	(A = 4; **B = 6**)

Now let the amount of A in X increase from 12 to 13, 14 and so on. As it does so, the amount of B in X must decrease accordingly. So, while X = 20, the ratio between A and B within X can change from 12 : 8 to 13 : 7, 14 : 6 and so on. What it cannot do is change so that there is more B than A in X, for in that case X would lose its distinctive qualitative character.

If the ratio between X and Y themselves alters, the ratio in each one between A and B remains subject to variation, provided that the preponderant quality continues to exceed the other in quantity. So if we now let X = 10 and Y = 20, A and B can still stand in a variety of different ratios within each:

X = 10	Y = 20
(**A = 9**; B = 1)	(A = 7; **B = 13**)
(or: **A = 8**; B = 2)	(or: A = 8; **B = 12**)
(or: **A = 7**; B = 3)	(or: A = 9; **B = 11**)

Note that the two sides of the major inverse ratio – X and Y – are thus both dependent on and independent of one another. They are dependent on one another insofar as they are the sides of that major ratio, since a change in the quantum of one directly brings about a change in that of the other. Yet each side is itself an inverse ratio between qualities, and so a "whole" within itself, and in this sense is "independent" of the other (SL 328 [ll. 10-12] / LS 422, ll. 10-12]).

The relation between the two sides, X and Y, however, has three further levels of complexity. The first is generated by the *qualities*, A and B, that are contained in each side. As we have seen, the unity of these qualities constitutes the (qualitative) substrate that is indifferent to the difference in quantity between X and Y. For this reason A and B are found together in both X and Y. The fact that A and B form a continuous *unvarying* substrate also means, however, that their combined quantum must remain the same as X and Y change their quanta. Yet not only is the unity of the two qualities invariant, but each quality is *itself* invariant as a moment of that invariant unity. This means, therefore, that, just as the sum of the two quanta of the qualities must remain the same, so too must the quantum of *each* invariant quality remain constant. Each quality, however, continues from X into Y (and vice versa) and so must preserve the quantum it has in the two sides taken together. That is to say, the total amount of A, considered as continuing from one side into the other, must remain the same, even though the amount of A in X and in Y, considered separately, may vary. As Hegel puts the point, "each of the two qualities taken singly for itself likewise remains the same sum which is the indifference; it continues from one side into the other without being restricted by the quantitative limit that is thereby posited in it" (that is, by the fact that it has one quantum in X and another in Y) (SL 329 / LS 423). So, if A increases in X, not only must B decrease in X to preserve the overall quantum of X, but *A* must also decrease in *Y* to remain the same sum across the two. (In this sense, the two sides, X and Y, are not really "independent" of one another after all.)

The numbers in the examples given above already reflect this complexity, since the overall amounts of A and B remain constant as the specific amounts in X and in Y change. There is, however, a second level of complexity in the relation between X and Y that is generated precisely by the fact that these *specific* amounts of A and B can change while their overall amounts remain constant.

We know that the quanta of X and Y can vary in inverse proportion to one another, while preserving the preponderance of A or B in each one and thus the "quality" of X and Y themselves. It also clear that the amount of A and B can vary within X and Y while preserving the "quality" of the latter. (This is the case in the examples above.) It is also clear, however, that changes in A and B can change the preponderance of either in X and Y and thereby change the "quality" of the latter: for, as the amount of A in X *decreases* and that of B

increases, so the amount of A in Y *increases* and that of B decreases, leading eventually to the switching of the "qualities" of X and Y. These "qualities", determined by the preponderance of A or B in the two sides, are thus vulnerable to alteration through merely quantitative change (as, indeed, qualities have been to a greater or lesser degree throughout the logic of measure).

The third level of complexity is encountered when the preponderance of A over B shifts from X to Y *and* in the process there is an overall increase in the quantum of Y and decrease in that of X. In this case, the idea of an inverse ratio of, or between, inverse ratios is fully realized, because the increase of Y at the expense of X coincides with the increase of A within Y at the expense of B.

INDIFFERENCE AND ITS *DASEIN*

Earlier in the logical development of measure, the latter took the form of a direct ratio between direct ratios (exemplified by the ratio between densities). Now measure takes the more complex form of an *inverse* ratio between *inverse* ratios. As we have just seen, this is a consequence of the fact that measure, or the unity of quality and quantity, is now a relation between two moments that are permeated by continuous, *indifferent* being.

These two moments – X and Y– differ quantitatively, but in such a way that they also differ qualitatively (because they *negate* one another directly in their inverse ratio). The unity of their two qualities, A and B, however, constitutes the indifferent substrate that is present in *both* moments. This means in turn that the qualitative difference between X and Y itself turns out to be a quantitative one: for, in sharing the same qualities, each differs from the other, and has a distinctive character and "quality", only by containing *more* of one quality and *less* of the other. Hegel puts the point like this: "these sides themselves thus continue into each other also according to their qualitative determinations; each of the qualities relates to itself in the other and is present in each of the two sides, only in a different quantum" (SL 328 / LS 422). The quantitative *difference* between the sides thus coincides with the indifferent *continuity* between them that consists in "the self-sameness of the qualities in each of the two unities".

Note that the "self-sameness" of quality here differs significantly from that found in the nodal line (see SL 324 / LS 418). In that line, the "continuing of the qualitative" is the principal constituent of the substrate of change; now, however, quality is explicitly combined with quantity in the substrate, since the latter consists in the sum of two quanta, as well as in the unity of their qualities.[14] Furthermore, in the line there is a definite difference between the changing moments and the unchanging, continuous substrate itself (despite the "ambiguity" of the latter). (At least, this is the case prior to the second paragraph of 1.3.3.A.) The changing moments are varying quantities and the various

measures and their qualities that are introduced by those quantities; the continuous substrate, on the other hand, consists in the unchanging qualities of the two elements that combine to form each different measure. The moments and the substrate coincide insofar as the former are states *of* the latter; but there is nonetheless a qualitative difference between them, insofar as the qualities of the enduring elements differ from those of the measures they form.

By contrast, in the inverse ratio we are now considering there is no longer a definite difference between the qualities of the moments and those of the substrate. Each moment or side of the ratio does, indeed, have its own "preponderant" quality – A or B – but otherwise each contains both qualities in the same way. This unity of the two qualities in turn constitutes the substrate of the moments. The qualitative dimension of each *moment* – X or Y – is thus supplied directly by the whole *substrate* itself, and (as Hegel puts it) each moment thereby contains "the indifference" within itself (SL 328 / LS 422). In the nodal line, the qualities, A and B, that form the substrate unite to form different moments with the qualities C, D, E and so on. In the current inverse ratio, by contrast, the different moments or sides consist simply in the unity *of* A and B, that is, in the substrate itself.

What gives each moment or side its distinctive character or "quality", therefore, is not just one of the qualities, but the quantum of each quality that is present in it: whether it contains more or less of A or B. The two moments are also distinguished by the total quantum or amount of units that each contains. Such *quantitative* differences, however, are all that separates the two moments and all that sets them apart from their substrate. This substrate consists in the sum of the two quanta, as well as the unity of their two qualities, but as the latter it is present in each of the two sides or moments of the ratio: as Hegel puts it, "it is immanent in all its determinations and in them remains in unity with itself and undisturbed by them" (SL 328 / LS 422). The two moments are thus, qualitatively, not only not different from one another, but also not different from that substrate itself. All that distinguishes them from their common substrate, therefore, is the fact that they differ *from one another* in quantity.

Differences in quantity are, however, precisely those to which the substrate is *indifferent*, and indifference, as we know, is ambiguous: it is both one with and different from its moments. The fact that the moments differ from the substrate only through their quantity thus reflects the fact that they are themselves both identical to *and* different from, or external to, that substrate. The moments are qualitatively *identical* to the substrate, since each is the same unity of qualities as the substrate itself. Accordingly, the moments differ from the substrate only insofar as they differ quantitatively from one another. Yet insofar as the moments differ only quantitatively, they are a matter of indifference to the substrate and in that respect are *external* to and *different* from the latter.

Note that since the indifferent substrate is present in, and as, the two sides of the ratio, indifference, as it is here conceived, actually entails a difference-*cum*-identity between two aspects, or forms, of *itself*. First, indifference has what Hegel calls its "*determinate being*" (*Dasein*) in the "totality" of different moments for which it supplies the substrate. Each of these moments – X and Y – consists in a changing quantum (in inverse ratio to that of the other) together with the two qualities concerned. As we have seen, however, the difference in "quality" between the moments is itself merely a difference in quantity: a difference of more or less. The *Dasein* of indifference thus in fact consists in a *quantitative* difference to which indifference is, precisely, *indifferent* and *external*.[15] At the same time, indifference is identical with its *Dasein*, insofar as it is present *in* the latter in the form of the unity of qualities that is common to both sides of the inverse ratio. Hegel draws attention to the identity *and* the difference between indifference and its *Dasein*, when he writes that the moments constitutive of the latter are "themselves the implicit [*ansichseiend*] totality of indifference" but are also "borne [*getragen*] *by* it as their unity" (SL 328 / LS 422, emphasis added).

Second, besides having its own *Dasein*, indifference is what it is *in itself*. As such, however, it is the unity *of* the determinations contained in its *Dasein* and so in that respect is not simply external to and different from the latter. In Hegel's own words, indifference is "in itself [*an sich*] the whole of the determinations of being which are resolved into this unity". Specifically, it is the sum of the two different quanta and also the unity of their two qualities. As this sum and this unity, however, indifference remains *different* from the moments that comprise its *Dasein*: the sum of the two quanta is distinct from either one taken separately, and the qualitative unity present in both sides of the inverse ratio is indifferent, and thus external, to the quantitative difference between them. Yet indifference is also *not* just different from those moments, since it is, precisely, the qualitative unity that is present *in* both of them.

Indifference thus has a complex twofold relation to the moments to which it is indifferent. This twofold relation is, however, *integral to the very idea of indifference*. Since the latter is quite indifferent to such moments, it does not mark itself off from them and so is not qualitatively distinct from them. In this respect, therefore, it is *one* with them and present *in* them. Yet, since it is indifferent to the moments, it is also necessarily external to them and different from them. The being that continues with indifference through various differences is thus both identical to and different from those differences: it is present in them and yet is also their underlying substrate.

Since the substrate's indifference to the moments it supports is an indifference to moments with which it is essentially one, it is in that sense indifferent to its *own Dasein* or, as Hegel puts it, "*towards itself* as a developed determinacy" (SL 328 / LS 422). Moreover, the substrate's *indifference* to its own determinacy

is evident in the fact that there is no intrinsic connection between the latter and it: in Hegel's words, "the in itself [*Ansich*] of indifference and this its *determinate being* are unconnected". This determinate being is thus not fully and explicitly the substrate's own, since it is not determined by that substrate. This lack of intrinsic connection manifests itself in turn in the fact that the different moments supported by the substrate emerge in the latter "groundlessly" and "immediately". Accordingly, they take the form of contingent quanta for which no explanation can be found in the substrate itself. Why should X have 20 units and Y 10? Who knows? "The determinacies show themselves in the indifference in an immediate manner" and so are simply external quantitative differences (SL 328 / LS 422).[16]

Hegel describes this lack of intrinsic connection between the substrate and its *Dasein* as a "defect" (*Mangel*) of indifference as it is here conceived (SL 328 / LS 423). Yet this judgement by Hegel is itself an external one, made from the perspective of the later logics of essence and concept – a perspective to which indifference is rightly indifferent. In the paragraphs that follow, however, Hegel points to a further problem with indifference that is immanent in it. This is the fact that indifference is implicitly contradictory.

THE CONTRADICTION IN INDIFFERENCE

Hegel begins 1.3.3.B.2.γ by reviewing the logical structure we have been considering. Since this review provides what is, by his notorious standards, a relatively clear summary of that structure, it is worth quoting in full:

> The quantitative determinacy of the moments which are now *sides* of the relation constitutes the mode of their *subsistence*; through this indifference [*Gleichgültigkeit*] their *determinate being* is removed from the passing away of the qualitative. But they have their *intrinsic* [an sich *seiend*] subsistence, which differs from this their determinate being, in the fact that they are *in themselves* indifference [*Indifferenz*] itself, each being the unity of the two *qualities*, into which the qualitative moment divides itself. The difference between the two sides is restricted to this, that one quality is posited in the one side with a more and in the other with a less, and the other quality similarly but conversely. Hence each side is in itself [*an ihr*] the totality of indifference.
>
> —SL 329 / LS 423

This logical structure is made necessary by the logic of measure, and so should be found in the world around us. In the remark following 1.3.3.B, however, Hegel supplies only examples to which, he claims, this structure is wrongly applied, and so he does not himself give a positive example of the structure.

The following, however, can be regarded as such a positive example. Take a government budget in which money is to be spent on two different areas, such as health and education (which match A and B in the examples above), but assume that these two areas are in fact inseparable from one another, such that spending on one entails spending on the other. There will, therefore, be no clear qualitative difference between the respective parts of the budget (X and Y above), but the latter will be distinguished from one another only by the fact that more is spent on health in one part and more on education in the other. The precise proportions spent on health and education in each part of the budget may vary, but the difference between the parts will be preserved if health predominates in one and education in the other. Now assume that a fixed amount of money is set aside for spending on health and education together. In this case, the two parts of the budget will be in an inverse ratio to one another, and (for any given numerical value of X and Y) the same will be true of the two areas within each part. The overall structure of the budget will thus be that of an inverse ratio between sides that are themselves inverse ratios, just as the idea of indifference requires.

This example shows that the inverse ratio inherent in the idea of indifference is not a mere fiction but can be encountered in the world. Hegel goes on to argue, however, that this ratio implicitly contains a contradiction. When we render this contradiction explicit, the inverse ratio we have described changes its character; indeed, it actually destroys itself. In so doing, it also takes us to the verge of essence and so brings the whole doctrine of being to a close.

Hegel's argument in 1.3.3.B.3 might make one think that the ratio we have been considering itself proves, on further reflection, to be an impossibility. I do not think, however, that this is what speculative logic demonstrates. What it shows, rather, is that rendering explicit the contradiction implicit in the inverse ratio of inverse ratios takes us on to a *new* logical structure that can no longer be such a ratio. (The thought that we are taken to a new structure in the course of 1.3.3.B.3 is suggested by the closing words of 1.3.3.B.2. In the ratio we have been considering, Hegel writes, "the determinations come into immediate opposition, which *develops into* a contradiction that we must now consider" [SL 329 / LS 423, emphasis added].)

So what is the implicit contradiction in indifference that we now have to render explicit? It is the contradiction between the idea that two qualities can be inseparable and the idea that there can be more of one than of the other. In the inverse ratio between inverse ratios the two qualities, A and B, form a unity and so are both contained in each side – X and Y – of the major ratio. Yet these qualities also retain a certain independence from one another, which manifests itself in the fact that the amounts of the two qualities in each side are not equal: each side contains more of one quality than of the other. If, however, the two qualities are meant to be *inseparable*, then they cannot in fact retain their

independence and, consequently, there can no longer be more of one than the other in either side. This contradiction remains implicit in the ratio as we have conceived it. It becomes explicit, and destroys the ratio, however, when the two qualities are understood to be *strictly* and *absolutely* inseparable.

If the two qualities are strictly inseparable, Hegel maintains, there can be no point at which they diverge and are not bound together: where there is one quality, so, too, there is the other. The two are thus not just linked at certain points – by certain parts of themselves – but they coincide completely; or, as Hegel puts it, "*each reaches only as far as the other*" (SL 329 / LS 424). Of course, a quality may be said to reach "only as far as" its counterpart in the sense that it extends only to the point at which it stops and the other begins, that is, only as far as the limit placed on it *by* the other. What Hegel has in mind here, however, is a different thought: namely, that the two qualities share the same reach and so completely overlap.

Hegel points out, however, that two qualities would *not* completely overlap if there were more of one than the other. If the amount of one were to exceed that of the other, they would coincide up to a certain point and then diverge: for one of them "would have in its *more* an indifferent determinate being which the other would not have" (SL 329 / LS 424). To put the point differently: the being of one would not always bring with it the being of the other, because *no* being of the one would match the excess of the other. This, however, is at odds with the idea that the two qualities are strictly inseparable: for in that case, as Hegel puts it, "in their qualitative relation [*Beziehung*], each is only insofar as the other is".

In the inverse ratio between inverse ratios, each side indeed contains more of one quality than the other. Yet when the two qualities are taken to be strictly inseparable, it turns out that this can no longer happen: "on the basis of their *qualitative* relation, there can be no question of a *quantitative* difference or of a *more* of the one quality" (SL 329 / LS 424).[17] The amount of one quality in each side must, therefore, match that of the other and the two must be in "*equilibrium*" (*Gleichgewicht*).[18] Accordingly, if the amount of one should increase or decrease, "the other would likewise increase or decrease and in the same proportion". This means that the two qualities in a side no longer stand in an *inverse* ratio to one another, but stand in a direct ratio of 1 : 1. The new structure that emerges when the qualities are conceived as strictly inseparable is thus no longer an inverse ratio between *inverse ratios*.[19]

The two qualities thereby lose the (albeit imperfect) *independence* they previously enjoyed through the "more or less". In the inverse ratio, as conceived in 1.3.3.B.1, the qualities it contains form a unity, yet each quality retains a certain independence from the other insofar as there is *more* of one than the other in a given side. In the new structure, however, there is no longer any quantitative difference between the qualities and so they no longer retain any vestige of independence.

Hegel states that "in this equality of the two [. . .] neither is present, for their determinate being was supposed to rest only on the inequality of their quantum" (SL 330 / LS 424). This does not mean, however, that the two qualities disappear altogether from each side of the ratio. It means, rather, that each side now contains qualities that have no *distinct* existence and so are no more than moments *of their unity*. This unity as a whole can be increased or decreased in quantity, but there cannot be different quanta of its constituent moments.

This, however, undermines the qualitative difference between the two *sides* – X and Y – of the major ratio.[20] In the ratio between inverse ratios, the two sides have different quanta, but they are also distinguished by the fact that each contains a preponderance of one of the qualities. It is this preponderance of one quality over the other that confers a distinctive character or overall "quality" on each side. Such preponderance has, however, now been eliminated because the two qualities in each side have proven to coincide completely. Consequently, there is no longer anything to confer a distinctive "quality" on either side of the major ratio. All there now is, therefore, is the inverse ratio between two different *quanta* of the same union of qualities.

Indeed – though Hegel does not make this point explicitly himself – that ratio (and difference) between quanta must also disappear, for, in the logical structure we are considering, it is inseparable from the qualitative difference between the two sides. As we saw above (2: 342), the inverse ratio between two quanta, generated by indifference, makes a qualitative difference between them necessary, because each directly negates and sets a limit to the other: "restricted by the fixed limit of their sum", Hegel writes, they relate "negatively to one another, and this is now the qualitative determination [*qualitative Bestimmung*] in which they stand to one another" (SL 327 / LS 421). Now, however, there can no longer be any qualitative difference between the sides of the ratio. Accordingly, there can no longer be any *quantitative* difference and ratio between them governed "by the fixed limit of their sum", for such a difference requires there to be a qualitative difference, too. There can therefore only be that sum. The two sides, X and Y, thus disappear altogether, and all there is is a single "whole" (*Ganze*), consisting in the unity of two strictly inseparable qualities, whose overall quantum can be increased or decreased (SL 330 / LS 424).[21] This unity of qualities constitutes indifference itself at this stage of the *Logic*. The inverse ratio produced by indifference, as it is conceived at the start of 1.3.3.B, thus collapses through its own internal logic *into indifference*.

The result we have reached is, therefore, this: not only is neither side of the inverse ratio an inverse ratio itself, but there is in fact no longer any major inverse ratio at all because there is no longer any qualitative or quantitative difference between "sides". The whole complex structure generated by the idea of indifference has thus been undermined; and this has occurred simply because

we have taken seriously, and in that sense made explicit, the *inseparability* of qualities that is at heart of that structure.

The logic of indifference requires that it give rise to an inverse ratio of inverse ratios: the latter is not a fiction but a logical necessity. Yet when the qualities involved are conceived as strictly inseparable – as they must come to be – the ratio of ratios collapses and we are left with nothing but a unity of moments. The inseparability of the qualities is itself the element of indifference present in both sides of that ratio, so the latter is brought to a collapse by the very logic of indifference itself. As we shall see in the next chapter, this collapse of the ratio – and differences – that belong to indifference results not only in a qualitative-quantitative "whole" – that is, in *indifference* – but in the transition to the altogether new logical sphere of *essence* (*Wesen*).

CENTRIPETAL AND CENTRIFUGAL FORCE

In the remark following 1.3.3.B Hegel notes that a simplified version of the category of indifference, conceived as an inverse ratio, is (or was) employed in science to explain the elliptical movement of celestial bodies, especially the movement of the planets around the sun. The facts of the matter, Hegel insists, are not in dispute: as planets follow their elliptical orbits "their velocity accelerates as they approach perihelion" – the point on an orbit closest to the sun – "and decreases as they approach aphelion" – the point on an orbit furthest from the sun (SL 330 / LS 425). These facts were established through the "untiring diligence of observation" (specifically observation by Tycho Brahe) and brought under a "simple law" (by Johannes Kepler), and in this respect, Hegel states, "all that is legitimately required of a theory" – that is, a scientific theory – "has thus been provided".[22] Reflective understanding, however, did not regard bringing observed phenomena under a law to be sufficient and so sought to "explain" (*erklären*) both the phenomena and the law by grounding them on the interaction between forces. In particular, "a *centripetal* and a *centrifugal force* are assumed to be the qualitative moments of movement in a curved line" (SL 330 / LS 425). These forces are taken to stand in an inverse ratio to one another, similar to the one we have just examined, and this inverse ratio is held to explain the acceleration and deceleration of the planets. (The "explanation" is that planets accelerate as the centripetal force strengthens and decelerate as it weakens in face of the strengthening centrifugal force.)

Note that this ratio between forces differs slightly from the ratio described in 1.3.3.B because it involves "only two qualities in inverse ratio to one another, not two sides each of which would itself be the unity of the two and their inverse ratio". Yet, as in the inverse ratio of "indifference", one quality or force increases as the other decreases, and vice versa, and it is this changing

relation between them that "explains" the acceleration and deceleration of the planets (SL 330-1 / LS 425). In Hegel's view, however, the proponents of this "explanation" fail to consider the logical consequences of the idea that they invoke.

According to this idea, as Hegel conceives it, the qualities or forces concerned are not independent but are bound together in "opposition" to one another; that is, each has its identity precisely in pulling against the other. This means, however, that one can extend only as far as the other does and that the two are thus inseparably united in their opposition: where there is one, there is also the other. As a result, neither can *exceed* the other in magnitude or strength, for if it did, its "excess" (*Überschuß*) over the other would no longer have a counterpart in the other and so "would not be present [*vorhanden*]" itself (SL 331 / LS 426). Since neither can exceed the other, neither can increase as the other decreases, and so no acceleration or deceleration is possible. This is clearly a version of Hegel's argument in the main text: if the two qualities in an inverse ratio are strictly inseparable, then there can be no quantitative difference between them and neither can be "preponderant". This is not to deny that one quality can exceed the other if they are united but not in absolute strictness (as in the example of health and education spending given above). Hegel contends, however, that in the idea of the two forces that are meant to explain planetary movement, "each force has meaning only with respect to the other"; in that case neither force can exceed the other and so they cannot explain the acceleration and deceleration of the planets after all.

Hegel adds that if we assume that one force *can* exceed the other, we encounter a further problem: for eventually "the greater would gain absolute predominance and the smaller would vanish" (SL 331 / LS 426). This is because in their inverse ratio the greater or stronger one force becomes, the less or weaker the other becomes, and a point will therefore be reached at which the latter force "*would no longer be able*" to wrest the body concerned from the former. The weaker force will thus eventually be totally overpowered and disappear.[23]

Note, however, that in the story told by reflective understanding what Hegel has just described does not happen; rather, the weakness of the centripetal force at aphelion converts itself – somehow – "into a preponderant strength over the centrifugal force" (SL 332 / LS 427).[24] Indeed, without such "conversion" (*Umschlagen*), a planet, on this story, could not begin to accelerate as it moves toward perihelion. (Equally, the strength of the centripetal force at perihelion must convert itself into weakness, if the planet is to decelerate as it moves towards aphelion.) How such a conversion occurs is not made clear by the understanding. What is clear, however, is that for it to occur one force cannot be altogether eliminated by an increase in the other, but must be preserved "as decreasing" (*im Abnehmen*) alongside the one that increases. This in turn

requires a revision of the inverse ratio that the understanding uses to explain planetary motion: for it must now be conceived as a ratio, not just between the two forces concerned, but between two "sides" each of which contains both forces. These two sides, Hegel contends, must be the two *movements* from aphelion to perihelion and vice versa, and since the one works directly against the other, they must be in inverse ratio to one another. The two forces contained *in* each movement must, however, also stand in inverse ratio to one another: for in the movement of a planet from aphelion to perihelion, the centripetal force must increase in strength as its centrifugal counterpart weakens, and in the contrary movement towards aphelion the centrifugal force must increase at the expense of the centripetal force. If, therefore, the alternating acceleration and deceleration of the planets is to be explained by the understanding, the category it employs to do so must be that of an inverse ratio of inverse ratios (as in 1.3.3.B).[25]

Hegel immediately points out, however, that the problem highlighted in 1.3.3.B.3 now manifests itself within each side: for if the two forces within each movement are bound together in opposition to one another, and thus inseparably united, then, logically, neither can exceed the other (and so no acceleration or deceleration can take place).[26] The idea of an inverse ratio between forces that are inseparably bound together through "opposition" thus proves incapable of explaining the phenomenon of planetary motion: for it "destroys" the phenomenon itself by making it logically impossible (or it remains an "empty" theory that adds nothing significant to the understanding provided by Kepler's laws) (SL 330 / LS 425).[27]

It is important to stress that in this remark Hegel is not challenging the findings of natural science. He is criticizing the "uncritical use" of the final category of measure to try to explain those findings (SL 333 / LS 428). In so doing, he argues not only that the category is not needed by science in this case, but also that it actually distorts our understanding of the phenomenon concerned. This is not to say that the category itself is completely without value or coherence. As we have seen, it is made necessary logically by the concept of measure and more immediately by the concept of indifference. Furthermore, as I have also suggested, the category is quite coherent if the two qualities united in it are not conceived as absolutely inseparable and coextensive. The strict inseparability of the qualities is, indeed, implicit in the idea of their unity, and in that sense the category is implicitly contradictory. It is, however, only when the qualities are taken explicitly *as* strictly inseparable that the category undermines itself. It is in this way that reflective understanding conceives of centrifugal and centripetal force – or, at least, it is in this way that it should conceive of them, if it is to preserve the unified "nature of the matter" (SL 332 / LS 427) – and as a result, Hegel contends, those forces fail to explain the phenomenon they were intended to explain.[28]

HEGEL AND SPINOZA

Hegel concludes his remark by briefly comparing "*absolute indifference*" with Spinoza's substance. He maintains first that they are similar, since in both of them difference is undermined – the difference between quality and quantity in indifference, and between thought and extension in Spinoza. As he puts it, "in both all the determinations of being, just as in general every further concrete differentiation of thought and extension, etc., are posited as vanished" (SL 333 / LS 428). Yet he also insists that indifference is to be distinguished from Spinozan substance: for the differences that are undermined belong to it – they are the differences *of* indifference – whereas the differences undermined by Spinozan substance do not belong to the latter. Hegel seems to me to misread Spinoza at this point, but the contrast he draws is nonetheless instructive.[29]

For Hegel, Spinozan substance cannot be conceived without difference, but it is itself the negation and absence of all difference: it is that in which there is *no* difference or determinacy (and, therefore, no negation) at all. In Hegel's words, the sole determination of such substance is "the negative one that everything is absorbed in it" (SL 333 / LS 428-9). Differences as such – between attributes and between modes – are thus only for the *understanding* or "intellect" and in that sense are external to substance (though Hegel adds, without noting the potential complication, that understanding is itself a *mode* of substance). Hegel's interpretation of Spinoza appears to find support in the latter's definition of an attribute as "what the intellect *perceives* of a substance, as constituting its essence".[30] Yet other remarks in the *Ethics* make it clear that, for Spinoza, attributes belong to substance itself and express "the reality, *or* being of substance".[31] Spinozan substance does not, therefore, lack all internal difference, as Hegel contends, for the different attributes that the intellect "perceives" actually constitute, and express, what substance *is*.[32] Such substance is thus closer to indifference than Hegel recognizes.

For Hegel, however, the two are clearly distinct; yet they are also related to one another, since, with indifference, difference is "now *posited* as it is implicitly [*an sich*] in Spinoza, namely as *external* and thereby more precisely *quantitative*" (SL 333 / LS 429).[33] As we have just seen, difference is already "external" for Hegel's Spinoza, insofar as "differentiation falls to the intellect"; what remains merely implicit in his thought, however, is the idea that the difference accompanying substance is external quantitative difference. Such difference is posited explicitly *as* external and quantitative, Hegel tells us, "by the determination [*Bestimmung*] of substance as indifference". This in turn suggests the further thought, mentioned above, that is absent from Hegel's Spinoza: namely, that external quantitative difference belongs *to* indifference itself (in a way it does not belong to mere substance), that the difference posited by indifference is precisely *its* difference. For Hegel's Spinoza, all difference is

merely there “for us”, since substance is the “negative” in which everything is “absorbed” and all difference disappears. For Hegel, however, difference is built into indifference itself *as* that which is merely external and quantitative.

Difference belongs *to* indifference, insofar as the latter itself takes the form of an inverse ratio between two different sides. Yet such difference is merely quantitative, since one side of the ratio differs from the other only by containing *more* of quality A than quality B, or vice versa. Furthermore, this ratio is *external* to, rather than “immanent” in, indifference, since it is not determined or “specified” by the qualities that constitute the indifferent substrate present on both sides. Those qualities do not require one or both of the quanta in the ratio to be raised to a power (as in the realized measure), nor do they determine that they will combine in certain ratios rather than others (as in the nodal line), but they simply find themselves in a given ratio to one another. This ratio is, indeed, spawned by indifference, insofar as the latter makes an inverse ratio necessary at all, and in that sense it belongs to indifference. Yet in that ratio there is no intrinsic connection between quality and quantity, and so the particular quanta concerned are purely contingent and immediate. In indifference, therefore, as Hegel conceives it, “the difference of quantitative and qualitative determination falls apart” (SL 333 / LS 429).[34]

What becomes clear in Hegel’s contrast between indifference and Spinozan substance – if this was not clear already – is thus that differences emerge *from* indifference itself as external quantitative differences to which it is precisely *indifferent*. Indifference is thus not mere indeterminate being, but being that is indifferent to *its differences* – being that is in-*difference*. As we then discover in 1.3.3.B.3, indifference, conceived as the strict inseparability of qualities, proceeds to undermine the quantitative difference between qualities that emerges from indifference itself. In so doing it also undermines the difference between the two *sides* of the ratio that depend on that quantitative difference between qualities. Indifference, therefore, is not just indifferent to its differences, but it actively dissolves them; or rather (which is the same thing), the different sides of the inverse ratio dissolve themselves through the unity of qualities that is indifferently present in both of them. Indifference, then, is in truth a process: the process of producing differences to which it is indifferent and which reduce themselves, through the indifference they contain, to indifference itself.

CHAPTER FIFTEEN

From Measure to Essence

INDIFFERENCE AND ESSENCE

The logical development of measure, as Hegel presents it, is long and complex, and it is easy to lose sight of the overall pattern that emerges. Here, therefore, is a brief summary of the principal stages in measure's development.[1]

Measure begins as the simple *unity* of quality and quantity, that is, as the "specific quantum" (or range of quanta) to which something owes its defining quality (see 1.3.1.A). It then stands in *relation* to quantity, as the rule by which a quantum is measured or as the "specifying measure" that alters the quantum that is added to it (1.3.1.B.a-b). Following this, measure takes the form of the relation between two *measures*, that is, of "the *realized measure*", expressed, for example, by Kepler's third law of planetary motion, $s^3 = at^2$ (1.3.1.B.c). Then it becomes an explicit *unity* once again, as the real measure proves to be, not just the relation between, but the "combination" of, two measures (that are themselves ratios), for example, the combination of two densities (see 1.3.2.A.a). After this, measure proves to be a *series* of such combinations, in which one measure, X, combines with a Series of different measures, A, B, C (in 1.3.2.A.b); and it then mutates logically into a series of measures produced by the *same* two qualities, such as oxygen and nitrogen, that are bound together in an "exclusive" unity. This latter series takes the form of what Hegel calls a "nodal line of measure-relations" (1.3.2.B). In this nodal line the two qualities constitute a permanent "substrate" that is *indifferent* to the different quantities in which they combine. The logic of indifference then requires it to take the further form that we examined in the last chapter: that of two qualities whose quantities are in inverse ratio to one another (1.3.3.B). This, Hegel now argues, brings the development of measure, and thereby of being as a whole, to an end, for implicit in the thought of such indifference is the thought of a new sphere altogether: *essence*.

Hegel begins 1.3.3.C, however, by emphasizing that indifference itself "still belongs to the sphere of *being* because it is still determined as *indifferent* [*gleichgültig*]" and so "has difference in it as *external*, quantitative" (SL 333-4 / LS 429). Indifference, in other words, is, and remains, indifferent to – and so immediately different from – the quantitative difference and inverse ratio in which it is expressed. As we saw in the last chapter (2: 343), the two sides of the inverse ratio are qualitatively identical to the indifferent substrate, since each side is the same unity of qualities as that substrate. The two sides differ from one another, therefore, only quantitatively, that is, only insofar as there is more of quality A than B in one side and more of quality B than A in the other side. Since, however, the sides differ only quantitatively in this way, they are a matter of indifference to the substrate itself and in that respect are *external* to and *different* from the latter. Indifference – the substrate – is thus present in, and one with, the two sides of the ratio, but also at the same time indifferent to them and thereby different from them.

The immediate and determinate difference between "this" and "that" – a difference that belongs to *being* – thus continues to attach to indifference, since it *differs* from the quantitative difference to which it is indifferent. In Hegel's words, indifference is "the foundation [*Grundlage*], but at first only in the *one-sided determination* of *being-in-itself*, and consequently the differences, the quantitative difference and the inverse ratio of factors, are present in it as *external*" (SL 334 / LS 430).[2] Moreover, the ambiguity in indifference – its being one with *and* different from its difference – is not an accidental feature of it, but belongs to its logical structure: indifference is indifferently present *in* its difference – in the two different sides of the inverse ratio – but precisely by being indifferent *to* that difference (since the latter is quantitative) it also differs *from* it.

Yet, as we have seen, the logic of indifference also dissolves this lingering difference in its structure; or, in Hegel's words, the "self-sublation of the determination [*Bestimmung*] of indifference has already occurred" (SL 334 / LS 430). This is the case because the difference from which indifference differs undermines itself and collapses into indifference itself. The two sides of the inverse ratio share the same two qualities (which constitute the "indifference" in both of them), and one side differs from the other *only* because one quality is preponderant in it, whereas the other quality is preponderant in the other side. The "qualitative" difference between the two sides thus resides in a quantitative difference, a difference of *more* and *less*. If, however, the two qualities are taken to be strictly inseparable, then there cannot be more of one than the other in a given side. This in turn means that there cannot be any "qualitative" difference between the sides, or any quantitative difference or ratio between them, and the sides collapse into a "whole" that is just the indifferent unity of qualities present in a certain quantity (see 2: 352). That is to

say, as the difference between the two sides of the ratio undermines itself (or is undermined by the indifferent unity of qualities in each side), the difference between those sides and indifference is itself undermined.

The logic of indifference, which makes the inverse ratio and its quantitative difference necessary in the first place, thus dissolves that difference and with it the difference between the latter and indifference itself. In the process, however, indifference actually ceases being *indifference* as such: for there is no longer any difference *to* which it can be indifferent. Indifference turns out, therefore, to undermine and negate itself. Moreover, the unity or "whole" into which indifference collapses must be conceived as the explicit *result* of that process of self-undermining. Pure being (at the start of the *Logic*) cannot be conceived as a result, since it is to be thought, precisely, as pure and simple *being* (see e.g. 1: 125); by contrast, the unity into which indifference collapses has to be conceived as a result for the following reason. Unlike pure being, that unity is not empty and abstract but consists explicitly in the "sublatedness" of the differences that belong to indifference itself: it is the *un*-differentiatedness, or *in*-difference, that indifference necessarily proves to be. As such, it points back explicitly to the "self-sublation" of those differences that gives rise to it, and so in itself is the explicit result of that self-sublation. In collapsing into a unity, therefore, indifference does not remain simple indifference, but nor does it just disappear. It proves, rather, to be the result of its own process of self-undermining and self-negation. As Hegel puts it, it proves to be "the negative totality whose determinacies" (or differences) "have sublated themselves in themselves [*an ihnen selbst*] and therewith sublated this their fundamental one-sidedness, their being-in-itself [*Ansichsein*]" – that is, their difference from indifference, as well as its difference from them (SL 334 / LS 430).[3]

This "negative totality", together with the process of self-negation that leads to it, is what indifference "in fact is" (*in der Tat ist*) when its constituent qualities are taken to be strictly inseparable. Yet such indifference, insofar as it is still *indifference*, necessarily remains – implicitly – a substrate that differs from determinations to which it is indifferent. Indifference, as self-negating, thus points logically *beyond itself* to a form of being that is no longer indifference at all. As self-negating *indifference*, however, it cannot itself be or become that new form of being. The latter arises only when indifference is thought explicitly *as* that which is no longer indifference, that is, *as* sheer self-negation – the "simple and infinite negative relation to itself, the incompatibility of itself with itself" (SL 334 / LS 430). When this occurs, indifference mutates logically into *essence*. Essence is thus not something quite other than indifference, but the form of being that arises when thought renders explicit, or "posits", the negativity that indifference "in fact is".

Indifference thus undergoes a complex development in the course of 1.3.3. First, it is the ambiguous substrate we encountered in the nodal line, or the

"continuing of the qualitative" in the latter (see SL 324 / LS 418). Second, it is the inverse ratio between two sides, or more precisely the unity of qualities in those two sides. Third, it is the strict inseparability of those qualities that causes the inverse ratio to collapse and thereby undermines the very idea of "indifference". Fourth, it transforms itself into essence, albeit only insofar as thought posits it explicitly *as* sheer self-negation or "negativity", rather than indifference.

ESSENCE AS "NON-IMMEDIACY"

Essence, as it first emerges in the *Logic*, is not conceived by Hegel as possibility, identity, substance, or any of the usual candidates. It is conceived as the "negativity of itself", the negativity that renders fully explicit what "indifference" proves to be (SL 334 / LS 430).[4] Indifference, as we have seen, begins as the substrate of the qualitative and quantitative differences that it makes necessary, but then its internal logic undermines both those differences and its own difference from them. In that sense, indifference negates itself through simply being itself and proves to be "the incompatibility of itself with itself". Essence is then *explicitly self-negating* being – being that is never just itself, and never simply different from what it is not, but that always negates and undermines itself. As such, essence consists, not only (like indifference) in the self-negating of difference, but also in the self-negating of immediacy, of being as such. It is the sphere in which nothing is immediately what it is, not even essence, and in which therefore there is – at least initially – no longer being or immediacy in the form (or forms) we have encountered so far.[5] Yet, like indifference, essence is also the *unity* that results from this self-negating of difference and immediacy – a unity that, in the case of essence, consists in the *relation to self*, or "*resulting, infinite going-together with itself*", that coincides with self-negation (SL 335 / LS 431).[6] "Essence", therefore, is the name that Hegel gives to being that has a quite specific character: being, not just as quality, quantity or measure, but as the self-relating unity that arises through, and contains within itself, the self-negating and "disappearing" of immediate being. As Hegel writes, essence is "being which, through the sublation of being, is simple being with itself" (SL 335 / LS 431).[7]

Note that, in *essence* (as opposed to mere indifference), being is so thoroughly self-negating that it is not *first* immediate and *then* self-negating, but it is self-negating from the start. In essence, therefore, being or immediacy, including immediate difference, has already been lost: "being as such and the being or immediacy of the different determinacies have thereby vanished just as much as *being-in-itself*" (or mere "indifference") (SL 335 / LS 431).

This is not to deny that quality, quantity and measure belong to essence, but they do not belong to it (at least initially) in the form in which they appear in

the doctrine of being. In the latter they are related to one another, since quantity is a form of quality, and measure is the explicit unity of quantity and quality; but each also has an immediate character of its own that distinguishes it from the others. Quality constitutes the very being of something; quantity as such is the quality to which something is indifferent; and measure is that quantity thanks to which something has the specific quality that defines it (and to which the thing is thus not indifferent after all).

In essence, by contrast, the different forms of being are present (at first), not in their distinctive immediacy, but only as purely relative moments – moments that are reduced to mere illusion, and then "posited" anew, by the process of *negativity* that constitutes "essence" (see SL 342, 346-9 / LW 9, 15-19). As Hegel puts it, "instead of affirmative beings [*Seiende*], as in the whole sphere of being, they" – the determinations of being – "are now simply and solely *as posited* [*als Gesetzte*]", and each is thereby "*related* to its other and negation" (SL 334 / LS 431).[8] What, precisely, it means to think of the determinations of being as posited *by* essence's negativity will only become clear in the doctrine of essence itself, which is not our topic here. What is clear now, however, is that (initially) essence is the sphere in which *there is no longer any immediate being*. It is the sphere of what Peter Rohs calls "non-immediacy" (*Nicht-Unmittelbarkeit*).[9]

Hegel states in the *Encyclopaedia Logic* that the doctrine of essence is "the most difficult" part of logic, and its complexities are indeed formidable (EL 179 / 236 [§ 114 R]). To conclude my brief discussion of essence, I will consider one particular difficulty: the relation, at the start of the doctrine of essence, between essence and being.

Essence, as Hegel conceives it, has a twofold relation to the sphere of being from which it emerges. First, the sphere of essence is a *new* one in which – in contrast to the sphere of being – there is (initially) *no* being or immediacy. This clearly implies that the various forms of essence, of non-immediacy, will differ from the forms of being (quality, quantity and measure); yet it also leaves open the thought that being can continue to be what it is alongside essence.

Second, essence is made necessary by indifference and, accordingly, is another stage in the immanent logical development of being. As such, it is not so much a different sphere from that of being, but rather the non-immediacy that being *itself* proves to be. In this case, however, it is hard to see how being could continue to be itself alongside essence, because its "immediacy" would now appear, in retrospect, to be an illusion. But then how did being give rise to indifference and essence in the first place? Or are the latter, as products of an illusory immediacy, themselves mere illusions?

In Hegel's view, we must understand essence both as a new sphere, different from that of being, *and* as what being itself proves to be. It is not initially clear, however, how it is possible to do so. In the next section, therefore, I will try to clarify the relation between essence and being.

ESSENCE AND BEING

Being is thought, at the start of the *Logic*, "in its indeterminate immediacy" as pure being: it is purely and immediately itself and nothing else (SL 59 / LS 71). In the course of its logical development being then loses its *sheer* immediacy, as it proves to be a moment of becoming and then to be inseparable from non-being (in *Dasein*), and as further forms of being emerge that are bound to, or turn dialectically into, their counterparts or opposites. These later forms of being lack simple immediacy because they do not stand alone as simply and immediately themselves, but they are what they are in relation to another. Nonetheless, in their relatedness they also retain a certain immediacy, insofar as each retains an identity of its own. So, although there can be no "something" without an "other", and thus no something that is not itself other than another, there is still a clear difference between being "something" and being "other". Indeed, it is this difference – more specifically the fact that the other is quite *separate* from its counterpart – that requires the other to stand "in isolation, in relation to itself", and thereby to give rise to change (SL 91-2 / LS 113-14). As Hegel puts it, therefore, "in the field of the qualitative, differences in their sublatedness also retain immediate qualitative being relative to one another" (SL 96 / LS 119).

In the sphere of quantity immediacy is reduced, or "sublated", more thoroughly than in quality, since quantitative determinations do not just pass over *into* one another, but in certain cases one determination already contains its opposite within itself. Continuity is thus already *in itself* the continuity of the discrete, and discreteness is *in itself* continuous discreteness, so neither has an identity that is immediately its own.[10] In this respect, indeed, quantity anticipates the logical structure that emerges in the doctrine of essence. In the sphere of essence, however, a determination, such as the positive, incorporates its opposite, the negative, as something *negated*: the positive includes the negative as *excluded* from it (see SL 367-70 / LW 42-6). These categories are thus above all the negation of their own negation, and in that sense lack immediacy. In quantity, by contrast, categories such as continuity and discreteness are contained *affirmatively*, rather than negatively, in one another. Accordingly, they retain a relatively *immediate* sense, even in their lack of immediacy and independence. So, although quantity does indeed anticipate the sphere of essence, it remains, with quality, a form of being.[11]

In measure immediacy is undermined more profoundly than in either quality or quantity, since the latter themselves prove to be functions of one another, or, in Hegel's words from the *Encyclopaedia*, "each is only *through the mediation of* [*vermittels*] the other" (EL 173 / 229 [§ 111]). This is the case, to some degree, throughout the logic of measure, but especially so in the nodal line, in which the qualities of the constituent measures specify the quantitative

proportions in which those measures may combine, but quantitative changes in turn produce changes or "jumps" in the quality of the combinations (SL 318-20 / LS 410-13).

Indifference is then the form of being in which immediate difference proves to be thoroughly self-negating. To begin with, in the inverse ratio produced by indifference, there is no clear difference between qualitative and quantitative difference, since the qualitative difference between the sides of the ratio is merely one of "more" and "less" and so in fact is just a quantitative "qualitative" difference. Then, as we have seen, this latter difference negates itself and collapses into indifference, and in so doing undermines the idea of indifference itself since there is no longer any difference for it to be indifferent to.

Note, however, that at this point immediate difference does not simply disappear. It proves, rather, together with indifference, to *be* the negating and undermining of itself. When indifference is then "posited" by thought as no longer indifference but *explicitly self-negating* being – and as relating to itself in its utter "negativity" – it becomes "essence". Essence, for Hegel, is thus initially not the indifferent substrate of being, but rather "being or immediacy which, through self-negation, is mediation *with itself* and relation to itself" (EL 173 / 229 [§ 111]). Essence, in other words, is not something other than or "beneath" being, but being *itself* insofar as it has negated and lost its immediacy (and so ceased being *being* as such). In Hegel's words (once again), essence is "being which, through the sublation of being, is simple being with itself" (SL 335 / LS 431). For this reason, as we have noted, the necessary forms of being – quality, quantity and measure – are not simply absent from essence, but they are present (at first) as merely relative, non-immediate moments: moments that are reduced to illusion, and then posited, by the negativity that is essence.[12]

Essence thus stands in an ambiguous relation to being and its immediacy. On the one hand, it is clearly different from being: it is a new sphere governed by non-immediacy, rather than immediacy and immediate difference. On the other hand, it is what being *itself* proves to be – being insofar as it negates, indeed has negated, its own immediacy. To put these two aspects of essence together: essence is being itself, though no longer *as* "being". Let us consider this ambiguity more closely.

As we move through the doctrine of being the relation between its major divisions changes. Quantity is a further form of quality, but it is also immediately distinct from the latter, since it is the "quality" to which quality itself is indifferent: red remains red, even if it becomes brighter or more intense. Measure, however, is not a third form of being quite distinct from the first two, but it is simply their *unity*. Nonetheless, measure has a character and "immediacy" of its own that is not reducible to mere quality or quantity, and it gives rise to logical forms that differ in subtle but significant ways from those of quality and quantity as such.[13]

Note that measure is the immediate, "*affirmative*" (*seiend*) unity of the qualitative and the quantitative, as can be seen especially (but not only) in the specific quantum, which is some "immediate" quantum or number that is attached to some "immediate" quality (SL 285, 288 / LS 368, 371).[14] Measure thus still falls within the sphere of immediate being. In contrast to measure, essence can be conceived as the *negative* unity of quality and quantity, the unity in which measure's "immediacy as well as that of its moments disappears" (SL 285 / LS 368). Yet, like quality, quantity and measure, essence is still what *being itself* proves to be; it is not something completely other than and outside of being. Specifically, as Hegel writes in the doctrine of essence, "essence is the absolute negativity of being; it is being itself [. . .] that has sublated itself both as immediate being and as immediate negation" (SL 342 / LW 9).

Note, however, that essence is not what being is from the start, but what being eventually, *finally*, proves to be.[15] We could say that it is what being proves to be *in truth*, and in the doctrine of essence Hegel states explicitly that "the *truth* of *being* is *essence*" (SL 337 / LW 3). Yet we need to be careful in interpreting this statement, for it is not now true, at the end of the doctrine of being, that being was never immediate in the first place. As we have seen, being takes the form of quality, quantity and measure, each of which exhibits its own immediacy, and it is not now the case that all such immediacy is ultimately *not* immediate after all. It is only the case that being makes necessary specific forms of itself – namely, the nodal line and indifference – *in which* being's immediacy is negated or negates itself. In the nodal line quality, quantity and measure all prove to be *mediated* moments, since the two qualities determine their quantitative ratios, but quantitative change in turn gives rise to new measures with new qualities. In indifference qualitative and quantitative differences then collapse into indifference that thereby itself ceases to be indifference, since there is nothing for it to be indifferent to. In these two forms of measure, therefore, "all the determinations of being are sublated" (SL 334 / LS 430). This does not mean, however, that all being whatsoever now proves to be "sublated" and non-immediate, because the nodal line and indifference are not the ultimate conditions of all quality and quantity. They are two forms of measure among others, and only in *them* is being's immediacy seriously undermined.

Indifference is the most developed form of measure and of being itself, and essence, as we have seen, renders explicit what indifference "in fact is" (SL 334 / LS 430). Essence is thus the sphere of *non*-immediacy that is made necessary by the whole logical development of being, and in that sense it can be understood as the "truth of being". The discovery that being must take the form of essence does not, however, show that *all* being is "ultimately" or "in truth" non-immediate, nor does it change what it is to be being as such. Being, *qua being*, continues to comprise quality, quantity and measure, and it will always do so, since these forms of being belong through logical necessity to being. Moreover,

even after being becomes essence, these forms in themselves still exhibit a certain immediacy: so being "something" still entails being immediately different from being "other". The emergence of indifference and then essence does not alter any of this: it does not change what it is *to be*. (In this sense, therefore, being can be said to continue to be itself alongside essence.) Nonetheless, *essence*, in which there is initially no immediacy but only self-negation or "negativity", proves to be the most developed form of being, or being in its "truth".[16]

So to repeat: essence is not something other than being, but it is what being *itself* proves to be. It is being, however, insofar as the latter is no longer immediate "being", but non-immediacy, negativity, relativity and mediation. As immediate being, being already exhibits the structures of negation and negativity: "something", for example, is "the *first negation of negation*, as simple self-relation in the form of being", and finitude is explicitly self-negating being (SL 89, 101 / LS 110, 126). Such negativity, however, does not deprive being of all immediacy and immediate difference: it does not undermine the very form of being *this*, rather than *that*. (Even indifference remains different, as *indifference*, from its self-negating differences.) The pure negativity that Hegel names "essence" does not, therefore, arise in being *qua being*, and so, when it does arise, being takes the form of non-immediacy that is *not* that of being as such. It is for this reason that essence is not just the most developed form, or "truth", of being, but also the *negation* of being (indeed what Hegel calls the "*first negation of being*") (SL 339 / LW 5). It is being itself in a form *that is no longer that of being itself*. In this latter respect, essence is a *new* sphere that we have not encountered before in the *Logic*, a sphere with different logical categories and relations that incorporate the categories of being (in a sublated form) but are not reducible to them.[17]

The fundamental ambiguity of essence thus derives from the fact that it is being in a new, wholly negative, form that is no longer that of being. As just noted, the discovery that being proves to be essence does not change what it means to be quality, quantity or measure: what it means *to be*. Yet *within* the new sphere of essence, and from the perspective of the latter, being is no longer immediate, but self-negating and non-immediate. From within the standpoint of essence, therefore, the "immediacy" of being and its forms is initially no more than an illusion (*Schein*), indeed an illusion that is projected by essence itself (SL 342-4 / LW 9-12).[18]

Yet this is not the end of the story: for the negativity of essence in fact reconstitutes or "posits" immediate being, first as merely mediated immediacy, but then as genuine immediacy – immediacy in the explicit form (or forms) that we have been examining in this study. Essence itself, in other words, makes necessary the explicit *existence* of quality, quantity and measure.[19] In this way essence proves, ultimately, not just to be essence, but to be the *unity* of essence

and being. This unity in turn takes the further forms of "actuality", "substance" and, eventually, the "concept" – which is the subject of the third book of the *Logic*.[20] In the course of the latter, the concept then develops into self-determining reason, or the "Idea" – which, as we saw in volume 1 (90), encompasses *all* preceding forms of being – and the Idea itself finally proves to be nature. The overall lesson of the *Logic*, therefore, is that being proves to be, not just being and not just essence, but reason in the form of *nature* – nature that leads logically to self-consciousness, or what Hegel calls *spirit* (*Geist*).[21]

CHAPTER SIXTEEN

Conclusion

Hegel argues that in the modern age of freedom and critique philosophy must itself be free and self-critical and so must avoid beginning from principles or assertions that are just assumed to be true. Philosophy, in other words, must be systematically (though not historically and hermeneutically) *presuppositionless*. Accordingly, it must begin with indeterminate being and then proceed by simply rendering explicit what is implicit in such being and subsequent categories. In this way, Hegel thinks, philosophy will unfold what is immanent in, and made necessary by, being itself, and so free, presuppositionless thought will at the same time be necessary and "scientific".

I noted in volume 1, however, that, for many of Hegel's critics, despite his talk of thinking "without presuppositions", his philosophy is in fact guided throughout by all manner of hidden assumptions (1: 50). I also stated that the merits or otherwise of this charge cannot be properly assessed until we consider the way speculative logic actually develops. We have now seen how it develops – up to the end of the doctrine of being – and so readers can determine for themselves whether such logic is based on hidden assumptions after all. My view is that it is not, and I have tried to show this by examining each category and explaining its logical transition into its successor in detail. On rare occasions I have suggested that Hegel's own presentation of a particular logical derivation is not as immanent as it should be, but in these cases I have indicated what the properly immanent derivation should be.[1] I have endeavoured to demonstrate, therefore, that the whole of the logic of being can be understood as the immanent, presuppositionless unfolding of what is implicit in, and made necessary by, pure being. It is now for readers to judge to what extent, if at all, this demonstration has been successful.

Of course, even if it is not successful, and if speculative logic is not as immanent as Hegel and commentators such as myself contend, that logic can

still contain insights of significance and value. In Hegel's view, however, what can *justify* such insights, and recommend them over alternatives, can ultimately only be the immanent, presuppositionless way in which they are derived. It is a matter of no small importance, therefore, to establish whether, or to what degree, speculative logic is in fact non-question-begging, immanent and necessary.

If, as I argue, it is non-question-begging and does develop immanently, then what we discover in the course of the *Logic* – and specifically in the doctrine of being that we have been considering – are two things. First, we discover the most basic categories of *thought* and how they are to be conceived. These categories guide our thought and action throughout our adult lives, but they are often only "instinctively and unconsciously" at work within us or conceived in an abstract manner (SL 19 / LS 19). Speculative logic thus shows us how to conceive properly categories with which we are already familiar, but which we do not usually reflect on. It shows us how to understand very simple categories, such as "something", and more complex ones, such as "measure", and it explains why we think in terms of specific categories at all. Why do we invoke the idea of infinity or think of a thing as *one* among *many*? And why do we think in terms of quantity? Why do we assume that things have size or that they can be counted? Speculative logic – and in particular the doctrine of being – provides answers to these questions by deriving the categories concerned from the sheer being of thought itself.

Second, we discover in the course of the *Logic* the most basic forms or ways of *being*, since speculative logic is also a metaphysics. The doctrine of being thus discloses what it is to *be* something, to *be* finite, to *be* a number and to *be* a measure. Not every commentator on Hegel's logic understands it in this way to be metaphysical, and to disclose fundamental aspects of being, throughout its course. Brady Bowman, for example, thinks that Hegel's account of finitude represents only "the attempt to conceive determinateness as arising through the mutual restriction (negation) of otherwise affirmative qualities", an attempt that *fails* and so leads Hegel to "take up" the category of infinity.[2] In my view, however, Hegel does not "attempt" and "fail" to do anything in his account of finitude, nor is his main focus on any such attempt and its failure. He sets out, rather, what it is to *be* finite, and he shows that finitude itself, as a way of being that belongs to every something, makes infinity necessary (see 1: 227-31). He then explains what it is to *be* infinite (in its different senses), and so on.[3]

Speculative logic does not tell us what it is to be an object in nature – to be a spatio-temporal entity – nor does it explain what it is to be a conscious, moral, social and historical subject. We have to wait until the (speculative) philosophies of nature and spirit to learn about these concrete ways of being-in-the-world. In this respect, therefore, logic gives us only a partial picture of what there is. Yet the logical structures or categories it unfolds are ways of being

that are exhibited *by* things in nature and self-conscious historical beings such as ourselves. The tree in my garden is *something*, *finite* and *one* among many, and it has a certain size or *quantum*. Speculative logic discloses what each of these ways of being involves, and in that sense it tells us about the tree and about other things in the world (and in the doctrine of the concept it goes further and discloses in purely logical terms what it is to be a *living* organism).[4]

In the doctrine of being the logic of measure in particular examines several logical structures that underlie significant phenomena in nature. The specifying measure, for example, is exemplified by specific heat, and the two higher forms of the realized measure are exemplified by laws that govern motion, namely Galileo's law of fall and Kepler's third law of planetary motion. Critics may argue that Hegel tailors his account of such measures to the natural phenomena, but in fact it is the other way round. Hegel derives the logical structure of each measure in an immanent manner from the one that precedes it, and then points to the natural phenomenon that exemplifies the measure. In this way, he does not derive the phenomena themselves from the nature of being – that can be done, if at all, only in the philosophy of nature[5] – but he shows that the measures underlying those phenomena are immanent in being itself. He thereby goes at least part of the way to proving that Galileo's and Kepler's laws are not just constructions of the human mind, but being's *own* laws – laws that being gives itself and that thus express being's *autonomy* (or *auto-nomy*).

There is no doubt that Hegel's derivation of the categories of thought and ways of being in the doctrine of being is often difficult to comprehend, and that his text poses considerable challenges to even the most sympathetic reader. In this study, however, I have attempted both to explain how, logically, the categories are derived and to shed light on Hegel's often forbidding sentences and paragraphs. I have also indicated various ways in which Hegel's conception of certain categories can be brought to bear on the thought of other philosophers. So, for example, I have highlighted the similarities and differences between Hegel and Plato on the one and the many, and between Hegel and Frege on number; and I have also considered the merits of (and some problems with) Hegel's account of Kant's antinomies. Categories, as Hegel conceives them, are complex and also dynamic (insofar as they lead logically to further categories); but they nonetheless have definite, intelligible logical structures, and this enables the student of Hegel's *Logic* to compare what he says about a specific category with the competing, or maybe complementary, claims of other thinkers. One of the aims of this study is to provide such comparisons in the hope that others may take them further.

The main aim of this study, however, is simply to help readers understand and assess the details of Hegel's logic of being. Many important issues are raised by the prefaces and introductory material at the beginning of the *Logic*, and I have explored some of these in volume 1. In the preface to the *Phenomenology*,

however, Hegel insists that “the matter [*Sache*] is not exhausted by stating it as an *aim*, but by *carrying it out* [*in ihrer* Ausführung]” (PS 2 / 5). The project of speculative logic is carried out in the detailed derivation of the categories, but this derivation is dense and difficult and can leave readers bewildered, in which case they may miss many of the insights provided by it. This study aims to guide readers through the thickets of Hegel’s doctrine of being in the hope of bringing those insights to light. The study will not be, and is not intended to be, the last word on Hegel’s logic of being. I hope, though, that it will enable others, who hitherto have struggled with Hegel’s *Logic*, or shied away from it altogether, to discover some of its riches, and even to enjoy being challenged by it, as I along with many colleagues and students have been fortunate to do.

NOTES

Chapter 1

1. See 1: 304.
2. See SL 166 (l. 34) / LS 211 (ll. 2-3). In the *Encyclopaedia Logic* Hegel notes that continuous and discrete magnitude should not be conceived as distinct "species" of quantity, if that means the determination of one does not belong to the other (EL 160 / 212-13 [§ 100 R]).
3. Degrees form a continuous "scale", but a degree is neither a discrete nor a continuous magnitude in the sense outlined here, since it neither contains, nor is divisible into, different degrees, but is a "*simple determinacy*" (SL 183-5 / LS 232-4). See 2: 144.
4. See SL 166 / LS 210: "the determinacy [*Bestimmtheit*] immanent to it".
5. Yet see this chapter, note 33.
6. See also EL 160 / 212 [§ 100].
7. Hegel's argument that there must be discrete magnitude thus does not support the thesis of Kant's second antinomy (see 1: 345).
8. Di Giovanni translates "stetig" as "continuous".
9. In this sense, Hegel remarks, the one is "sublated" in discrete magnitude (SL 167 [ll. 17-18] / LS 211 [l. 33]).
10. See Houlgate (2014), 20.
11. McTaggart (1910), 48, and Winfield (2012), 137.
12. See SL 168 / LS 213: "a determination of itself against *other* quanta".
13. See 1: 258, 303, and Biard et al. (1981), 149 (ll. 16-17).
14. See SL 168 (ll. 23-5) / LS 213 (ll. 22-4).
15. SL 167 / LS 212: "the limit [. . .] is also, as one, *related to itself* [*auf sich bezogen*]".
16. See SL 132 / LS 166, and 1: 257-9.
17. In the *Encyclopaedia Logic* Hegel states that "the quantum is the *determinate being* [*Dasein*] of quantity" (EL 161 / 214 [§ 101 A]). It is, however, more precisely quantity in the form of a limited something (*Etwas*).

18. McTaggart (1910), 48.
19. See Hartmann (1999), 100.
20. See SL 168 (ll. 23-5) / LS 213 (ll. 20-2). Discreteness is, of course, explicit in continuity from the start, since the latter is the continuity-*of-the-discrete* (see 1: 303). Yet discreteness is more to the fore, and so more explicit, in discrete magnitude, than in continuous magnitude, and so, by comparison, the many are merely "implicit" in the latter.
21. See SL 155 (ll. 13-16) / LS 196 (ll. 3-6).
22. Recall that qualitative ones cannot differentiate themselves from one another in this way, because they have no internal "plurality". They establish their distinct identities, therefore, by *repelling* one another, which paradoxically *unites* them into one "one of attraction" (see 1: 272-6).
23. Di Giovanni and Miller both translate "Einheit" as "unit" (see SLM 203).
24. Note that, since both continuous and discrete magnitude can be divided into quanta, and quanta are determinate only as numbers, both kinds of magnitude can be assigned numbers and so be counted or measured (see SL 170-1 / LS 216, and EL 163 / 216 [§ 102 A]). – Kant's conception of number is similar to Hegel's, except that he understands number to summarize "the successive addition" of units *in time* (CPR B 182), whereas Hegel derives number from quantity without reference to time. See Houlgate (2014), 26.
25. See also EL 161 / 214 [§ 102], and Martin (2012), 112.
26. Hartmann (1999), 103.
27. Aristotle (1984), 2: 1669 [1056b25-6]; see also 1: 373 [220a27].
28. By thinking of "one" as a number, rather than just a bare unit, one can also add units to its "amount" and so produce greater numbers – though such numbers can also be thought as the products of "aggregations, repetitions of the one" considered as a bare unit (VGPW 1: 237).
29. I am not aware of any discussion by Hegel of the number zero, but he does distinguish at one point between "a qualitative nil" and the "nil of the quantum" (see SL 221 / LS 282). Negative numbers, by the way, can be generated by further, albeit notional, subtraction of units from zero, or by conceiving of numbers in "opposition" to positive natural numbers (see SL 371-4 / LW 46-50).
30. An exception to this may be the speed of light, which Hegel describes as "absolute velocity" (EN 87 / 112 [§ 275 A]). See also Cox and Forshaw (2009), 104: "the cosmic speed limit".
31. See SL 168 (ll. 23-5) / LS 213 (ll. 22-4).
32. See SL 169 / LS 214: "It is rightly said of amount that it *consists* of the many, for the ones are not in it as sublated but *are* in it [sind *in ihr*]".
33. A number is thus in this respect a "discrete magnitude" (SL 169 / LS 213), even though the units in continuous magnitude (that is, into which it can be divided) can also be counted. A number's discreteness, however, exceeds that found in discrete magnitude as such, since the latter does not "consist of" wholly discrete units (see SL 166 / LS 210).
34. Burbidge (2006), 48. At this stage in the argument, this "definite set" could – conceivably – be infinite (if a number were to contain all the units there could be). As we will see later, however, Hegel rejects the idea of an infinitely large (or infinitely small) number or quantum as incoherent. See 2: 162.

35. SL 169 (ll. 7-8) / LS 214 (ll. 2-3).
36. Aristotle (1984), 1: 8, 353 [4b24, 207b7].
37. See this chapter, note 33 for the subtle difference there is, for Hegel, between a number and a discrete magnitude as such.
38. See SL 169 / LS 214: the ones in an amount "*are* in it, only posited with the excluding limit to which they are indifferent".
39. SL 171 / LS 217: "number is an external aggregate [. . .] without any inner connectedness". On the process of "numbering", through which units are counted together, externally, as one number, see SL 172 / LS 217-18, and 2: 26-7.
40. Di Giovanni and Miller translate "Anzahl" in this passage as "number" (see SLM 204).
41. Winfield (2012), 138.
42. Di Giovanni translates "Beziehung" as "connecting reference" rather than "relation".
43. See VGPW 1: 237: number is "determinate, but without opposition, indifferent".
44. Numbers, as here conceived, might thus be regarded as "real" numbers, though at this stage in Hegel's account they are only a subset of the latter, namely the cardinal natural numbers (2, 3, 4 and so on). (Thanks to Christopher Yeomans for this thought.)
45. See SL 171 (ll. 14-16) / LS 216 (ll. 33-5).
46. To my knowledge, Hegel does not state this explicitly, but it is implied by the logic he has been tracing.
47. See Carlson (2007), 161: "numbers do not add themselves". As we shall see, however, numbers form a continuity by themselves (rather than by external addition), insofar as concrete quanta, which are determinate only as numbers, necessarily *change* into other quanta. See SL 189-90 / LS 239-40, and 2: 153-5.
48. See SL 208, 272-3 / LS 265-6, 351-3.
49. See SL 211-12 / LS 270: "infinite series that cannot be summed [. . .] contain a higher kind of infinity than do those that can be summed, namely an incommensurability, or the impossibility of displaying as a quantum, even in the form of a fraction, the quantitative ratio that they contain; but the *form of the series* as such that they have contains the same determination of bad infinity that is present in the series that can be summed".
50. Ferrarin (2001), 135.
51. Boyer (1959), 20, 40; Klein (1992), 46, and Aristotle (1984), 2: 1611 [1020a12].
52. Klein (1992), 7.
53. Plato (1982), 130-1 [56d-e].
54. Shapiro (2000), 59.
55. See Plato (1993), 58, 62 [101c, 104a-c].
56. Klein (1992), 71. Dougal Blyth sees a prima facie contradiction in the thought that numbers are both forms (and "so essentially unities") and "multiplicities of units". He resolves the contradiction, however, by claiming that the "form numbers are essentially ordinals, not cardinals", and that the cardinal numbers – which are collections of units – are "mathematical images of the form numbers" (Blyth [2000], 30). By contrast, David Gallop argues that in the *Phaedo* Plato does not draw a clear distinction between numbers and their forms (Plato [1993], 97, 100 [notes to 101c

and 104a]). In this chapter I avoid taking sides on this issue by stating that, for Plato, numbers "are, or at least are derived from", forms.

57. Shapiro (2000), 69.
58. Klein (1992), 103. See Aristotle (1984), 2: 1665 [1054a7]: "the number is a number of particular things".
59. Aristotle (1984), 2: 1709 [1081a20]. See also 2: 1664 [1053a30]: "number is a plurality of units".
60. See Klein (1992), 104-5; Lear (1982), 183-4, and e.g. Aristotle (1984), 2: 1704 [1078a22-5].
61. Klein (1992), 53, 191, and Aristotle (1984), 2: 1663 [1052b20-5].
62. Klein (1992), 54. For Hegel, we recall, the one that is the "source", or logical origin, of number is the qualitative one, not the number one. The latter is itself derived from numbers in the full sense by subtraction (see 2: 12-13).
63. Klein (1992), 56.
64. Klein (1992), 106. Klein has "odd" instead of "even".
65. Aristotle (1984), 2: 1710 [1082a22-6], and Klein (1992), 106.
66. See Aristotle (1984), 1: 8 [4b24-31].
67. Klein (1992), 108.
68. See Aristotle (1984), 2: 1719 [1088a7-12], and Annas (1975), 99-100.
69. Note, by the way, that although, for Aristotle, we comprehend a number as *one* because we do our counting over one and the same thing, there is no number "one" itself. The smallest number, in Aristotle's view, is two. See Aristotle (1984), 1: 373, 2: 1669, 1715, 1719 [220a27, 1056b25-6, 1085b10, 1088a6]. Julia Annas argues, however, that, despite his official view, Aristotle, like Plato, *uses* 1 as a number (Annas (1975), 100).

Chapter 2

1. See this volume, chapters 3-5.
2. See also SL 178, 702-3 / LS 225, LB 244-5.
3. See also EL 162 / 215 [§ 102 R]. Klein takes the same point to be made by Plato: "addition and also subtraction are only an extension of counting" (Klein [1992], 20).
4. See EL 162 / 215 [§ 102 R]: "A *species* of calculation, however, is the counting together of what are no longer mere ones but already numbers".
5. Note that Kant's use of the term "analytic" to mean that one concept is "contained" in another through the "principle of contradiction" should be distinguished from its use to describe the "analytic unity" of concepts or consciousness, *under which* other representations (including those that are not connected with them "analytically") can be brought (see e.g. CPR B 12, 105, 133n).
6. For a summary of the different senses of "analytic" in Hegel, and how they differ from Kant's conception of the term in his idea of an "analytic" judgement, see this volume, chapter 3, note 68.
7. See SL 221, 235 / LS 283, 301, and 2: 210 ff.
8. See also EL 163 / 215 [§ 102 R]: "there cannot be more than these three kinds of calculation".

Chapter 3

1. Parsons (1995), 182.
2. Currie (1982), 42.
3. Dummett (1991), 111.
4. Moore (2012), 196-7.
5. Dummett (1991) 82.
6. Moore (2012), 215-16.
7. Russell (1961), 701.
8. Moore (2012), 216, and FR 203 / GGA xvi.
9. See e.g. Brandom (2002), chapters 6-9.
10. Dummett (1991), 1.
11. See Dummett (1991), 2, 7.
12. See also GA 122, 138 [§§ 90, 109]. Translations of the *Foundations* are my own or, when the relevant text is included in *The Frege Reader* as here, by Michael Beaney (occasionally amended).
13. Dummett (1991), 1.
14. Dummett (1991), 4, 9.
15. See FR 253 / B 59 [Russell to Frege, 16 June 1902]. See also Shapiro (2000), 114-16; Weiner (2004), 123-6, and 2: 84-6.
16. See Dummett (1991), 6-7, and Weiner (2004), 125-6.
17. Russell (1919), 11. See also Shapiro (2000), 115-16.
18. Weiner (1990), ix, 1; see also GA 122 [§ 90]: "a gapless [*lückenlos*] chain of inference". Frege suggests, indeed, that "arithmetic would be simply a further developed logic" (GA 119 [§ 87]; see also FR 138 / FB 11).
19. For Hegel, by contrast, gravity is inherent in matter itself. See EN 46 / 62 [§ 262 R], and Houlgate (2005a), 133-4.
20. On the differences between Hegel and Kant concerning "7 + 5 = 12", see 2: 27-30.
21. See Allison (2004), 90.
22. See FR 69-70 / BS 19-20 [§ 11], and Weiner (2004), 44-7.
23. Frege actually describes "all whales are mammals" as a "proposition" (*Satz*) here, rather than a "judgement" (*Urteil*). For the purposes of this discussion, however, I do not distinguish between the two. For the distinction drawn in the *Begriffsschrift* between a "Satz" (or "judgeable content") and judgement (in which that content is affirmed as true), see FR 52-5 / BS 1-5 [§§ 2-4], and Weiner (2004), 31. In later texts, Frege uses the word "Satz" to refer to a series of sounds (though not just "*any* series of sounds") that has a "sense", and when this sense is what he earlier called a "judgeable content" – a content that can be true or false, as opposed to the content of e.g. a command – he names it a "thought" (*Gedanke*). Judgement is then understood to be the recognition or affirmation of this thought as true, and an assertion is the explicit communication of this judgement to others. See FR 149 / B 35 [Frege to Husserl, 24 May 1891]; FR 239 / L 54; FR 327-30 / G 33-6, and Currie (1982), 112-14.
24. Note that in the *Begriffsschrift* Frege uses a Gothic "a" instead of "x" in this context. On the difference in the *Begriffsschrift* between Gothic letters that stand for variables and Latin letters that do so, see Currie (1982), 25, and Kanterian (2012), 49, 162-8.

25. See FR 152-3 / ÜSB 24-5, and FR 149-50 / B 35 [Frege to Husserl, 24 May 1891].
26. Moore (2012), 201.
27. See Weiner (1990), 84, and Weiner (2004), 16, 19.
28. Note that Frege thinks geometry is synthetic in this sense; see GA 121 [§ 89].
29. See BS 26 [§ 13], and FR 378.
30. See BS 26, 43, 47, 50 [§§ 14, 17, 19, 21], and FR 380-1.
31. Recall that, for Hegel, the word "*Anzahl*" denotes the amount *in* a number, rather than a number as such (SL 169 / LS 214).
32. See Weiner (1990), 42, 45.
33. Beaney translates "Zeichen" as "symbol" (FR 114).
34. Weiner (1990), 103; see also 87.
35. Resnik (1980), 180. See also FR 314 / LM 100.
36. Moore (2012), 215.
37. See FR 51 / BS xiv [Preface]: "Arithmetic [. . .] was the starting point of the train of thought that led me to my *Begriffsschrift*".
38. Moore (2012), 213.
39. See GA 91 [§ 58]: "the number [. . .] is neither something sensible nor a property of an external thing".
40. See also FR 233 / L 45-6, and Kanterian (2012), 3.
41. Moore (2012), 213.
42. See also FR 228 / L 38: "like ethics, logic can also be called a normative science".
43. See Weiner (2004), 148, and Kanterian (2012), 32.
44. See also Weiner (1990), 59.
45. On the relation between presuppositionless logic and empirical cognition, see also EL 28-9, 31, 37 / 47, 49-50, 57-8 [§§ 6, 7 R, 12 R]; Houlgate (2005a), 115-21, and 1: 105.
46. The third section of Hegel's philosophy of subjective spirit is entitled "Psychology", but this discipline is a rational, largely a priori one that is different from empirical psychology as Hegel conceives it (see EPM 165 ff. / 229 ff. [§ 440 ff.]).
47. See e.g. FR 202 / GGA xv: logical laws lay down "how one should think".
48. See Weiner (2004), 149.
49. Weiner (2004), 149; see also 15, 65, and Kanterian (2012), 33.
50. See GA 107 [§ 74], and 2: 132-4.
51. Di Giovanni translates "an ihr selbst" as "Internally".
52. See Houlgate (2011), 147-52.
53. See SL 19-20 / LS 20: "at no stage of the development should any thought determination or reflection occur that does not directly emerge at this stage and does not proceed in it from the preceding determinations".
54. See 1: 54.
55. See Houlgate (2006), 29-53, and 1: 41-3. On the possibility of beginning logic with *nothing*, see 1: 143.
56. See EL 6, 75, 258-9 / 17, 105, 334 [Preface to 2nd edn, §§ 36 A, 182 A].
57. See GA 119, 122 [§§ 87, 90], and Weiner (1990), x.
58. See also Weiner (1990), 41.
59. Weiner (1990), xii, emphasis added.
60. See Weiner (2004), 102.

61. See e.g. GA 32 [§ 7] on John Stuart Mill's "prejudice [*Vorurteil*] that all knowledge is empirical".
62. Dummett (1991), 25. See also FR 310-11 / LM 94-6.
63. Weiner (2004), 21.
64. See Weiner (2004), 19.
65. Edward Kanterian points out, however, that the basic laws of logic cannot themselves be analytic, since they are not derived from other more basic laws (see Kanterian [2012], 22).
66. Weiner (2004), 20, 118.
67. See 2: 27-30, and this chapter, note 68.
68. To pull Hegel's different conceptions of "analytic" together, one can say the following. 1) The basic operations of *arithmetic*, namely numbering and addition, are "wholly analytic in nature" because they entail the completely *external* combining of two or more units or numbers (SL 173 / LS 219). This sense of "analytic" is directly opposed to Kant's idea that in an analytic judgement one concept is contained *in* another. 2) The mathematical equation "7 + 5 = 12" is an "analytic" judgement in the sense that "5 + 7 and 12 are absolutely the very same content" (SL 704 / LB 246). This sense of "analytic" is close to Kant's, but differs from the latter because, in Hegel's view, "5 + 7" and "12" are not *concepts*. 3) Speculative philosophy, and more specifically speculative *logic*, is "analytic" insofar as it proceeds through "the mere *positing* of what is already contained [*enthalten*] in a concept" (EL 141 / 188 [§ 88 R]). This sense of "analytic" comes still closer to Kant's, since it does involve concepts. Yet, for Hegel, concepts or categories are, strictly speaking, not already "contained" in those that precede them, but a new category *emerges* from another as thought renders explicit what is implicit in the latter. This sense of "analytic" is thus different from Kant's after all. (As I have noted, the "analytic" character of logic, for Hegel, is in fact inseparable from a moment of "synthesis", and this, of course, makes Hegel's understanding of "analytic" in this case even more different from Kant's. On the fusion of the "analytic" and "synthetic" in speculative method, see also SL 741 / LB 291, and 1: 89.)

Chapter 4

1. See 1: 63.
2. See SL 50 / LS 62, and Houlgate (2006), 61.
3. See SL 356-8 / LW 27-9, and Houlgate (2011), 142-50.
4. See e.g. Williams (2017), 52, 94.
5. See also Williams (2017), 116, 261-3.
6. EL 59 / 84 [§ 24 A2]. See also EL 246 / 318 [§ 167], and SL 517, 588 / LB 16, 104.
7. See VGPW 2: 235, and SL 67 / LS 82.
8. Hegel also sees value in practising "abstract thinking" in preparation for speculative logic (see SL 35-6 / LS 41-2).
9. See SL 620 / LB 143; FR 58-9 / BS 7-9 [§ 6], and Weiner (2004), 25-6, 32-3. Note that, for Hegel, the hypothetical judgement that forms the major premise of the hypothetical syllogism – namely, "if A is, so is B" – expresses the relation between a ground A and its consequent B (SL 576-7, 620 / LB 90-1, 143). This, however, need

not be the case for Frege. In his view, that major premise could be e.g. "if (or given that) the sun is shining, then 3 x 7 = 21", in which the sun's shining is not the ground of the equation (see FR 56 / BS 5-6 [§ 5], and Weiner [2004], 33). Frege thus appears to reduce the premise "if A, *then* B" to "if A *and* B" (or even just "A *and* B" without the "if") (see FR 63 / BS 12-13 [§ 7], and Kanterian [2012], 92).

10. See Houlgate (2006), 37-9, and 1: 49-51.
11. See Parsons (1995), 182.
12. See Weiner (1990), 70, and Weiner (2004), 15.
13. See also Weiner (2004), 21. It was noted on 2: 43 that, for Frege, "the laws of number are [. . .] not really applicable to external things". Nonetheless, "they are certainly applicable to judgements that are made about things in the external world", and it is this idea, I think, that Frege has in mind here (see GA 119 [§ 87]).
14. Dummett (1991), 46. See also Weiner (1990), 80.
15. See 1: 4-7, and 2: 46.
16. See 2: 32.
17. See e.g. SL 746 / LB 298, where Hegel describes the form of "triplicity" as "the merely superficial, external side of cognition".
18. See Moore (2012), 216.
19. See also FR 244 / L 61: "freeing us from the fetters of language".
20. See also Weiner (2004), 26-8.
21. In his *Begriffsschrift* Frege does not describe a function explicitly as "incomplete", but he suggests as much by talking of it as having one or more "argument-places" (see FR 67 / BS 16 [§ 9].)
22. Another – non-mathematical – conceptual content that is not judgeable, and so cannot be true or false, is the simple idea "house" (see FR 53 / BS 2 [§ 2]). Yet the latter is not a concept in the full Fregean sense, since, for him, "a concept is a function whose value is always a truth-value" (FR 139 / FB 11). The proper concept "house" is thus "( ) is a house" (see 2: 73-4).
23. In *Begriffsschrift* Frege maintains that the symbol for identity "signifies the circumstance that two names have *the same content*" (rather than designate the same *object*) (FR 64 / BS 14 [§ 8], emphasis added). On the difference between Frege's early and later conceptions of identity, see Weiner (2004), 77-8, 88-90.
24. Weiner (2004), 39-40.
25. See FR 146 / FB 20: "we call such functions of two arguments relations".
26. Weiner (2004), 24.
27. See e.g. Aristotle (1984), 1: 28 [18a13-16], and Weiner (2004), 42.
28. See Weiner (2004), 40.
29. Weiner (2004), 22-4.
30. Weiner (2004), 24.
31. Weiner (2004), 48. See also GA 138 [§ 108]: "the inference from n to $(n + 1)$ [. . .] is based on general logical modes of inference".
32. Weiner (2004), 40.
33. Weiner (1990), xii-xiii. See also Weiner (2004), 111.
34. See Weiner (2004), 37 ff., 109, and FR 68 / BS 18 [§ 10].
35. See FR 6, and Weiner (2004), 56, 71. See also GA 23 [Intro.]: "The distinction between concept and object must be kept in mind". Note, however, that the

distinction between a function and an argument in the *Begriffsschrift* is (in Currie's words) "a conventional one" – since one can treat different aspects of a judgeable content or of an equation as "function" or as "argument" – whereas the later distinction between concept and object is a logical-ontological one. See Currie (1982), 21.

36. See Weiner (2004), 71.
37. GA 90 [§ 57], and FR 140 / FB 13. See also Currie (1982), 62.
38. For Hegel, both categories and the objects they constitute are in some respect "unequal" to themselves, since they are inextricably bound to their negations. One could argue, therefore, that, in Frege's terminology, they are both in that respect "incomplete". See also Houlgate (1986), 162-3.
39. Resnik (1980), 192, emphasis added.
40. See Weiner (2004), 104. Object-words can, of course, be proper names, such as "Julius Caesar", and in this case they do not require a definite article.
41. Beaney translates the phrase "ein Gegenstand kommt nicht wiederholt vor" as "an object does not occur anywhere else" (FR 102).
42. Frege does not deny that a judgement can relate two (or more) objects to one another and state, for example, that "Jupiter is bigger than Mars"; but such a judgement can also be understood to subsume the singular object Jupiter under the concept "being bigger than Mars". (For a similar case, see FR 183 / ÜBG 49.) – Nor does Frege deny that one can make judgements about collective entities, such as "the Roman people". Any such entity, however, will itself be a singular individual (as is indicated by the definite article). Conversely, Frege claims, entities that are more obviously singular, such as *this* tree in my garden, can themselves be regarded as collective entities: "we consider every physical body to be a whole, a system, consisting of parts" (B 70 [Frege to Russell, 27 July 1902]. – Note that the lines quoted in the main text show that Frege is prepared to employ the distinction between subject and predicate, even though he insists in the *Begriffsschrift* that "a distinction between *subject* and *predicate* finds *no place* in my representation of a judgement" (FR 53 / BS 2 [§ 3]).
43. See FR 69 / BS 19 [§ 11], and Weiner (2004), 44.
44. See e.g. FR 10-12.
45. See also this chapter, note 42.
46. OFG 33 / ÜGG 371. See also Weiner (2004), 110.
47. For Frege's use of the word "something" (*etwas*) in this context, see e.g. FR 174 / ASB 27, and FR 186 / ÜBG 53. Beaney translates "etwas" as "anything".
48. See Weiner (2004), 78-9. Similarly, the words "something" and "it" are said to be "indefinite" or "indeterminate" (*unbestimmt*) (LA 168-9).
49. In Frege's text the whole phrase is italicised.
50. In some renderings the quantifier takes the form "for all x" (see FR 12); but the point we have just made remains the same.
51. "Alle und jede Etwas sind geradesogut *Diese*, als sie auch Andere sind". Di Giovanni mistranslates this sentence as: "every something is just as good a 'this' as any other". – Note the difference, for Hegel, between the "this" and the thought of "all individuals". The former is meant to refer to a single individual, but fails to do so on its own; the latter, by contrast, refers to all single *individuals* together (see 2: 76).

52. See Weiner (2004), 103-4.
53. See 2: 38. Frege points out that it is possible for a *concept*, not just an object, to fall *under* another concept, namely under "a second-level concept" – a "relationship that is not to be confused with that of subordination" (of one first-level concept, such as "whale", to another first-level concept, such as "mammal") (GA 87 [§ 53]). So, for example, if we say that "'there is at least one square root of 4', we are saying something, not about (say) the definite number 2, nor about -2, but about a concept, *square root of 4*; viz. that it is not empty" (FR 187-8 / ÜBG 54). In so doing we bring the first-level concept "square root of 4" under another, second-level, concept "not being empty", or rather "being realized [*erfüllt*]". (Frege suggests that one could also say that a first-level concept "falls *into* [*in*]" a second-level concept [FR 189 / ÜBG 56].) If, however, we express the same thought by saying that "the concept *square root of 4* is realized", then "the first six words form the proper name of an object" and we say something about this object, not about a concept. It remains the case, therefore, that whenever we talk about "*the* concept X", we convert the concept concerned into an object and so "miss" (*verfehlen*) our thought (FR 188, 192 / ÜBG 54, 59). Or, to put it another way, whenever we say of "*the* concept X" that it is "a concept easily attained", or simply that it is "a concept", our statement is *false*, since we say this of an object that, as such, is precisely *not* a concept (see Weiner [2004], 108).
54. Note, therefore, that a "real" (*eigentlich*) contradiction arises, for Kant, only when two assertions are made and are assumed to be true, one of which must be true and the other *false* (e.g. "every body smells good" *and* "*not* every body smells good"). A real contradiction does not arise, however, if two opposing assertions are made, both of which could be false (e.g. "every body smells good" and "every body smells bad") (see CPR B 531, and 1: 337).
55. On Frege's conception of the objectivity of logic, see 2: 43-5.
56. See also FR 141 / FB 14, and FR 178 / ASB 32.
57. See vol. 1, chapters 8-12.
58. See PS 355-63 / 385-94, and Houlgate (2013), 162-5.
59. Strictly speaking, not all the categories in the *Logic* constitute "objects" or "somethings", but some, such as true infinity, constitute *processes* (see 1: 244-5). Nonetheless, all are understood to be constitutive of some aspect of being itself. – Recall that the subject-matter of Hegel's logic is simply *being* and the categories that are inherent in it. Yet, as we saw in volume 1, we retain, throughout such logic, a reflective awareness that being is the initial form taken by pure *thought*, and that in examining being, and so doing ontology, we are also examining thought, and so doing logic. At the start of Hegel's logic, therefore, we have a double consciousness of what is before us: we know it to be pure *being*, but also to be being and thought together (see 1: 118).
60. See FR 189 / ÜBG 56, and 2: 43-5.
61. See GA 100, 137 [§§ 68, 107], and Weiner (2004), 119.
62. Functions that are not concepts include, for example, mathematical ones such as "2. $( )^3$ + ( )". The values of this function for the arguments 1 and 2 would thus not be truth-values, but the numbers 3 and 18 (see FR 133-4 / FB 5-6).
63. Currie (1982), 68. Thanks, too, to Michael Beaney for clarification on this point.
64. See FR 253 / B 59 [Russell to Frege, 16 June 1902]; FR 7-8, 279-80, and Weiner (2004), 123-5.

65. See also FR 173 / ASB 26.
66. As Frege explains in “Function and Concept”, relations (*Beziehungen*) are functions with two (or more) arguments. As such, they fall on the side of concepts, rather than objects (FR 146 / FB 20). This in turn allows a judgement that relates two or more objects to one another to be converted into a judgement that subsumes an object under a concept (see this chapter, note 42).
67. See Weiner (2004), 63: “The introduction of the notion of extension of a concept leads to disaster”.
68. The problem is thus not just due to Frege’s introduction of Law V in *Basic Laws*, as Frege believed (see FR 7-8, 279-89, and Weiner (2004), 116, 123-5).
69. Weiner (2004), 62-3.
70. See, for example, Dummett (1991), 210, and Brandom (2002), 258 ff.
71. Dummett (1991), 111. See also GA 92, 94 [§§ 60, 62].
72. See Brandom (2002), 238-9, and FR 13.
73. See 1: 38, and Houlgate (2006), 12-23.
74. See 1: 70-4; Houlgate (1986), 141-66, and Houlgate (2006), 93-8.
75. On how we hold at bay the familiar meanings of words at the start of the *Logic*, see 1: 75-6, and Houlgate (2006), 79-88.
76. See Moore (2012), 197. Beaney inserts the word “ordinary” before “linguistic means of expression”, but this word has no equivalent in Frege’s German text.

Chapter 5

1. Dummett (1991), 82.
2. Aristotle (1984), 1: 373 [220a27].
3. Di Giovanni translates “Recht” as “privilege” (which is the usual translation for “Vorrecht”).
4. See Euclid (1908), 2: 277 [*Elements* VII, Def. 2]: “A number is a multitude composed of units”.
5. On Hegel’s understanding of the number 1, see 2: 12-13.
6. Hume (1955), 171; see GA 67 [§ 34].
7. Dummett (1991), 85.
8. Frege makes this claim about points in space, but in his view it applies to any things that are taken to be equal. All things (and “representations” of things) must be different in some way if they are “not to flow together into one” (GA 76 [§ 42]).
9. Leibniz (1996), 230 [2, xxvii].
10. Leibniz (1998), 269 [*Monadology* § 9]: “in nature there are never two beings that are perfectly alike”, and Leibniz (1996), 156 [2, xvi]: “‘number’ as a multitude of units”.
11. Leibniz (1998), 110 [Leibniz to Arnauld, 4 / 14 July 1686], emphasis added.
12. For evidence that Frege read Leibniz’s *New Essays*, see e.g. GA 46 [§ 17]. See also Kanterian (2012), 8.
13. Jevons (1913), 162. See also GA 69 [§ 36].
14. See GA 72 [§ 39].
15. See also FR 132 / FB 4.
16. See also LM 118-19.

17. See Dummett (1991), 86-7.
18. Leibniz (1998), 269 [*Monadology* §9], and Leibniz (1996), 290 [3, iii].
19. See SL 631 ff. / LB 156 ff.
20. Leibniz (1996), 290 [3, iii].
21. See 2: 83. For Hegel's account of the universal and singular as such, see SL 530 ff. / LB 33 ff.
22. One should bear in mind, however, that "ones" are not separate entities alongside things (see 1: 249). Finite things themselves prove to be ones, insofar as they exhibit being-for-self. *As ones*, however, they are no longer explicitly finite, or qualitatively different, but each is just the *one* that it is (and consequently one of many, and so on).
23. Dummett (1991), 82.
24. See also GA 57 [§ 26]: "number is also something objective".
25. Note that the argument here assumes that b is not identical to a.
26. See GA 51 [§ 22]: "In the same way, an object to which I can ascribe different numbers with the same right is also not the proper bearer of a number".
27. Indeed, a relation can be regarded as a kind of concept, since it is simply a function with two (or more) arguments (FR 146 / FB 20). See 2: 70.
28. For Hegel, too, the quantity of something can be determined *by us* in different ways. This, however, is due, not to the nature of number itself, but to the fact that any quantitative *standard* (*Maßstab*) we might use to measure the magnitude of a thing – feet or inches, pounds or kilos – will be external to the thing being measured and in that sense arbitrary. A thing can thus be judged to be 2 feet in length *or* 24 inches. See SL 289, 291-2 / LS 372, 375, and 2: 251-2.
29. See SL 180 / LS 228, and 2: 65-6.
30. See 2: 14: "Number, for Hegel, is thus a peculiar hybrid that is both intrinsic to being *and* a matter of indifference to, and so external to, being at the same time".
31. Though, for a moment, the difference between repulsion and attraction remains qualitative (see 1: 277, 282, 300).
32. See SL 185-6 / LS 235, and 2: 150-2.
33. Aristotle (1984), 2: 1719 [1088a7-12]. Note that this measure is different from the external quantitative "standard" mentioned in this chapter, note 28; nor is it a measure in a further Hegelian sense, namely a quantity that enables something to exhibit a certain quality (see 2: 246 ff.). The measure that Aristotle has in mind consists in the (or a) *quality* of things that, for him, enables them to be counted together.
34. See Klein (1992), 108: "We comprehend a number as *one* because we do our counting over one and the same thing".
35. Aristotle (1984), 2: 1716 [1085b35-6].
36. Spinoza (1995), 259.
37. Spinoza (1995), 104 [Ep12]. See GA 56 ff. [§ 26].
38. See FR 193 / ÜBG 60.
39. See e.g. Weiner (1990), 45, 84.
40. Dummett (1991), 111-12.
41. Dummett (1991), 112.
42. Dummett (1991), 111.
43. Dummett (1991), 114,

44. Hume (1888), 71 [1, iii, i], and GA 94-5 [§ 63].
45. "Eindeutig" here simply means "unambiguously", though Beaney translates it with good reason as "one-one" (see FR 110). I follow Beaney here (if not exactly) by rendering "eindeutig" as "one-to-one".
46. See Weiner (2004), 60-1.
47. Dummett (1991), 128.
48. Dummett (1991), 119.
49. For Frege, a relation between two or more objects can always be converted into the subsumption of one object under a concept (see vol. 2, chapter 4, note 42). Since concepts are understood by Frege to be (or to express) the properties of the objects that fall under them (see FR 189-90 / ÜBG 56), this suggests that the relation of one object, A, to another, B, can itself be understood as a *property* of A. This in turn brings Frege close to Leibniz, for whom all relations are properties of the things in relation (see Leibniz [1998], 59-60 [*Discourse* § 8], and Jolley [2005], 51). For Leibniz, however – in contrast to Frege – relational properties are just that, and cannot also be understood as genuine external relations *between* things. Things or "substances" do not, therefore, act on one another externally, but "nothing determines them except God alone" and their own "perfect spontaneity" (Leibniz [1998], 84-5 [*Discourse* § 32]).
50. Weiner (2004), 119.
51. See B 70-1 [Frege to Russell, 27 July 1902], and Currie (1982), 41: "an extension has an essential unity". In this respect, a Fregean extension (or class) is similar to a Hegelian number, which is both an aggregate and a unity (see SL 169 / LS 214) – though, of course, such a number, unlike an extension, is not a set of objects that "fall under" a concept.
52. See GA 99 [§ 67], and 2: 41-2.
53. In this case, the objects correlated with those under F will include the objects under G but also those under H, I, J, K and so on, and the objects correlated with those under G will include the objects under F as well as those under the other concepts.
54. Beaney has "directions" instead of "lines" for "Geraden".
55. See 2: 80-1.
56. Dummett (1991), 120, 122.
57. Dummett (1991), 122, emphasis added.
58. Dummett (1991), 122.
59. See GA 103 [§ 70]: "The individual pairs of correlated objects . . .".
60. See Dummett (1991), 122.
61. It remains the case, however, that, for Frege, "an object does not occur repeatedly" (GA 85 [§ 51]). To be identified or "defined", it must be recognized as the same one *again*; but in each case it is, precisely, the *same* object, since an object is always only *the* singular object it is.
62. So, although criteria of identity do not tell us precisely what their corresponding objects are, Dummett is right to suggest that they are an "essential aid" in the process of defining those objects (or their object-words). See Dummett (1991), 119.
63. Recall that in § 55 Frege does not *define* 0 or 1 (or any other number), but he only determines "the sense of the phrases 'the number 0 belongs to [*kommt zu*]', 'the number 1 belongs to'" (GA 89 [§ 56]). See 2: 113.

64. Indeed, to say that *nothing* falls under a concept amounts to saying that 0 *objects* fall under it. Numbers, however, are meant to be ascribed only to concepts, not to objects, so there is all the more reason not to use the word "nothing" here.
65. At this point one should, of course, recall that, for Frege, *the* concept under which no object can fall is itself an *object* (see 2: 80).
66. See 2: 49, and also vol. 1, chapters 8-10.
67. See SL 201-2 / LS 256, and 2: 162-3.
68. See e.g. FR 380-1 / BS 50 [§ 21], and FR 143 / FB 16.
69. Shapiro (2000), 111. In this case, the number belonging to "either identical to zero or identical to one" is *n*, and the number belonging to "identical to zero but *not* identical to one" is *m*. *M* is the number 1, since its concept is equivalent to "identical with 0" and only 0 falls under it, so *n* follows 1 in the number series. Accordingly, the appropriate number word to attach to *n* is "2".
70. See Winfield (2012), 138.
71. Numbers, for example, are both *unities* and aggregates of wholly *discrete* units, and so in that sense are not simply identical with themselves (see 2: 15, 18). Zero is contradictory, perhaps, insofar as it has an amount that comprises no units and so is not actually an *amount* after all (see 2: 13).
72. EL 1 / 11 [Preface to 1st edn]: "*proof*, i.e. the very thing that is quite indispensable for a scientific philosophy".
73. On the possibility of beginning logic with *nothing*, see 1: 143.
74. See e.g. EL 82 / 114 [§ 41 A1]: "thinking that is free is without presuppositions".
75. See e.g. EL 80, 124 / 112, 168 [§§ 39 R, 78 R].
76. Other features of speculative logic also require that it be systematically presuppositionless and begin with pure being, including the very nature of *beginning* itself (SL 50, 52 / LS 61-2, 64-5) and the fact that the aim of logic is to *discover*, and not just presuppose, the categories and laws of thought (SL 23 / LS 25). See vol. 1, chapter 3, *passim*.

Chapter 6

1. See 2: 18-19.
2. See SL 178 (l. 5) / LS 225 (l. 16).
3. See SL 182 (ll. 32-3) / LS 231 (ll. 32-4).
4. On SL 182 / LS 231 Hegel writes that "extensive quantum is distinguished from number only because in the latter the determinacy is explicitly posited as plurality", that is, as a specific amount. By implication, therefore, this is not the case in the extensive quantum or magnitude.
5. As we saw on 2: 14-15, the same discrete ones in the number constitute both its amount and its unity. Yet, since there is a "qualitative difference" between the latter two moments (SL 171 / LS 217), the discrete ones in a number constitute its amount only insofar as they do not constitute its unity (and vice versa). Accordingly, insofar as they constitute the *amount*, they are not "sublated" into moments of a unity or continuity but are wholly discrete, self-relating ones (see SL 169 / LS 214).
6. See Houlgate (2014), 23-4. – Hegel's argument here is subtly different from the argument Frege deploys against the Greek conception of number. Frege claims that,

if things could be made equal and be regarded as identical units, "one would no longer have things, but only one thing"; that is, they would "melt irretrievably into one" (GA 68, 75 [§§ 35, 41]). As I explained in vol. 2, chapter 5, however, this argument ignores the fact that quantitative units remain discrete, and so fall *outside* one another, even though they are identical and form a continuity (see 2: 94-5, 100-1). Hegel's claim that the units in extensive magnitude collapse into a "simple unity" might now seem to suggest that Frege is right after all. Yet the reason why such units form a simple unity is not the one Frege gives. It is that those units form a unity that is qualitatively *distinct* from an amount and so lacks explicit plurality. *Pace* Frege, identical units can form a continuity or unity while remaining many; when, however, the unity is explicitly *contrasted* with the plurality of an amount, it will lack the latter and be simple.

7. On the similarities and differences between Kant and Hegel with regard to extensive and intensive magnitude, see Houlgate (2014), 26-7.
8. Winfield (2012), 138, emphasis added.
9. Di Giovanni has "each many", instead of "each of the many", for "jedes der Vielen".
10. SL 183 / LS 232, and SLM 218.
11. See e.g. SL 184 / LS 233: "But the quantum has its determinacy as amount".
12. See SL 185 (ll. 21-3) / LS 235 (ll. 4-7).
13. By contrast, cardinal numbers do not by themselves form an explicit continuity or a new number but have to be added together externally to form such a number (see 2: 20).
14. On SL 187 / LS 237 Hegel states that not only the degree but "the number as such has its meaning only in the number series". This cannot mean, however, that every number, qua number, "derives" its meaning in that series (as di Giovanni puts it), since each cardinal number, as we have seen, is *"determinate-in-itself"* (SL 170 / LS 215). In my view, Hegel's statement should be taken to mean that cardinal numbers give rise to a series – through the addition or subtraction of one unit – but not that they first acquire meaning in that series.
15. Di Giovanni has "determined within".
16. See also SL 187 (ll. 19-20) / LS 237 (ll. 22-4), and Winfield (2012), 138. Similarly, if 0 begins a list of numbers it is the 1st, not the 0th, in that list.
17. See SL 187 (ll. 8-9) / LS 237 (ll. 8-10).
18. "Ton" can mean "sound", but Hegel has in mind the heat-induced expansion of ceramic objects.
19. See SL 167 / LS 212: "quantum – quantity as a determinate being [*Dasein*] and a something [*Etwas*]".
20. See vol. 1, chapter 14, note 12, and 2: 109-10. The logic of quantity thus begins with pure quantity by itself (SL 154 / LS 194).
21. See Hartmann (1999), 114.
22. See SL 189 / LS 240: "A quantum, according to its quality, is therefore posited in absolute continuity with its externality, with its otherness".
23. See WLS 162.
24. Johnson (1988), 46.
25. As we will see, a number relates to *itself* in the power to which it is raised (since the latter is its own product), but it does so in becoming *another* number: so $3^2 = 9$. The

number 3 is also contained in any higher number it becomes insofar as the latter will always be 3 + n (see 2: 19); that higher number, however, is not itself still 3.

26. An exception is found in the speed of light, which Hegel describes as "absolute velocity". The fact that this speed doesn't change is due, however, not to its being a quantum, but to the logical character of light, as Hegel conceives it, that is, to light's being "pure identity with itself" (EN 87 / 111-12 [§ 275 and A]).
27. I have said in this chapter that the concrete quantum is "expressed" by both a cardinal and an ordinal number. This form of language mirrors Hegel's when he says that the determinacy of a degree is "expressed by a *number*" (SL 183 / LS 232). Such language, however, is not meant to suggest that a concrete quantum is the *substrate* of the numbers in which it is expressed. The qualitative something that emerges in 1.2.2.B.b can be understood in this way, but the concrete quantum cannot (see 2: 153). The concrete quantum is what the number proves to be and so is itself a number: it is both a cardinal and an ordinal number at once. Yet, precisely because it is both at once, it is not reducible to either and so is not reducible to being a simple *number*. It is a new form of quantum that is a number but also *more than just a number* – a new quantitative "something" that is both extensive and intensive and that continues beyond itself as its number becomes another and another and so on. I have tried to capture this idea that the concrete quantum is, but is also more than just, a number by saying that it is "expressed" by one.
28. Numbers can, of course, form an endless series through addition or subtraction, but quantitative change makes such a series logically necessary.

Chapter 7

1. See SL 133, 169 / LS 168, 214; 1: 257-8, and 2: 15, 156-7.
2. On the endless succession of finite things, see SL 108 / LS 134, and 1: 227-8.
3. SL 191 / LS 242: "the quantitatively finite or the quantum as such".
4. See SL 190 (ll. 16-17) / LS 241 (ll. 20-3).
5. See SL 190 (ll. 20-2) / LS 241 (ll. 27-30).
6. Leibniz, too, sees a contradiction in the idea of "the largest of all numbers" (Leibniz [1998], 54 [*Discourse* § 1]). See also CPR B 459-60, where Kant rejects the idea of a quantitative "maximum".
7. See SL 202 (ll. 13-14) / LS 257 (l. 9).
8. SL 203 (ll. 2-3) / LS 258 (ll. 5-7).
9. See SL 203 (ll. 18-19) / LS 258 (ll. 28-9).
10. Recall that change belongs to the "quality" of being a concrete *quantum* (SL 189 / LS 240; see also 2: 153). This, however, does not yet make the quantum itself explicitly qualitative. It proves to be the latter only when it relates "infinitely" to itself in relating to another and thereby exhibits the explicit quality of *being-for-self*.
11. See EL 168 / 222 [§ 105]: "determinacy *that is for itself*" (fürsichseiende *Bestimmtheit*).
12. See EL 168 / 222 [§ 105]: the "quantum is quantitative *relation* [*Verhältnis*] – determinacy that is both an *immediate* quantum (the exponent), and *mediation* (namely the *relation* [*Beziehung*] of some quantum to another)".
13. As we noted in this volume, chapter 6, numbers can be changed externally (through addition or subtraction), or they can change of their own accord through being

concrete quanta that continue beyond themselves in other quanta (2: 154-5). In the example we are considering here what matters is only *that* the numbers can be changed, not how and why.

14. See EL 59, 166 / 85, 220 [§§ 24 A2, 104 A2]. Note that fractions are direct ratios between numbers (see SL 208-9 / LS 265-6). As Christian Martin points out, therefore, Hegel's logical derivation of the direct ratio expands the range of numbers to include not just the natural numbers but all the positive rational numbers (see Martin [2012], 118). Irrational numbers, which cannot be expressed in the form of a fraction, require the idea of the quantitative infinite progress in order to be conceived (see SL 211-12 / LS 270.)

Chapter 8

1. In this remark Hegel claims that the second antinomy "contained the opposition of qualitative finitude and infinity". In his earlier analysis of that antinomy, however, he understands it to set the two moments of pure *quantity* in a qualitative opposition to one another by treating them as quite distinct (see SL 157 / LS 198, and 1: 313).
2. See SL 156, 200 / LS 197, 254, and EN 29 / 42 [§ 254 R].
3. For detailed studies of the first antinomy, see Grier (2001), 183-94, and Allison (2004), 366-76.
4. In the Transcendental Aesthetic Kant describes space as "an infinite *given* magnitude" (CPR B 39). It is "infinite", however, not in the sense of a maximum, but in the sense that one can progress in any direction *without end* (see A 25: "boundlessness in the progress of intuition"). Kant's rejection of the idea of a quantitative maximum is thus not at odds with his conception of space in the Aesthetic.
5. See SL 191, 199 (ll. 8-9) / LS 242-3, 252 (ll. 37-9), and 2: 161.
6. For Hegel's critique of Kant's conception of infinity, see SL 206-7 / LS 263-4.
7. See Grier (2001), 184: "an infinite series is by definition a series that has no completion".
8. See Grier (2001), 185.
9. See Carlson (2007), 178: "The past is conceived as radically separate from the future. Hence, the very introduction of 'now' – a point in time – presupposes time's beginning".
10. See EN 15-16 / 26-7 [§ 247 A], and Winegar (2016), 97.
11. See Grier (2001), 190.
12. On the subtle difference between Hegel's speculative conception of quantity as continuous-but-divisible and the one-sided conception of continuity he extracts from the antithesis of the second antinomy, see 1: 359-60, 366-7.
13. The idea of infinite divisibility, with which Hegel resolves the second antinomy, is also a form of such infinite progress, but it is grounded in the genuinely speculative unity of continuity and discreteness.
14. See e.g. SL 163 / LS 206 on space.
15. If one were to claim that space itself can grow, then one would have to presuppose a further surrounding space (or "nothingness") *into* which it could grow – a space that would itself have to be there all at once.
16. Einstein (1960), 112, and Kennedy (2003), 148.

17. If space were, indeed, to prove to be finite yet unbounded, this would be an example of the way in which nature sets limits to quantitative change that are not intrinsic to quantity as such (see 2: 158).
18. See Winegar (2016), 97, and EN 36 / 50 [§ 258 A]: "it is therefore the process of actual things themselves which makes time".
19. Hegel does not himself speak of the true infinity of space; but he understands an "enclosing surface" to separate off a "*single* whole space", so one might be justified in regarding the latter as space in the form of being-for-self (see EN 31 / 45 [§ 256]).
20. For a more detailed account of space and time in Hegel's philosophy of nature, see Houlgate (2005a), 122-31.

Chapter 9

1. Note that the direct ratio is not the truly infinite quantum as it first emerges in the quantitative infinite progress, but the initial form taken by this quantum when its true infinity has been rendered explicit (see 2: 164-6).
2. On SL 272 / LS 351 Hegel describes the exponent as just a "quantum", but it is in fact "infinite" insofar as it is fixed. Note that Hegel's use of the term "exponent" differs from that current in mathematics. In the latter the exponent is the superscript number that indicates how many times a number is to be multiplied by itself: so 2^3, 3^4 and so on (see Burbidge [2006], 50).
3. See this volume, chapter 7, note 13.
4. See also 2: 148.
5. See also 2: 167.
6. See SL 273 / LS 353: "as exponent, therefore, this quotient is not posited as what it should be".
7. Two quanta stand in a kind of inverse ratio when their exponent is just their fixed sum. In this case, if one is e.g. 1, 2, 3 and the exponent is 10, the other is 9, 8, 7. This, however, is not an inverse ratio in the full sense, since the exponent is not the explicit unity of unity and amount. It is merely an amount, rather than a certain amount *of* the unit, or the unit *multiplied* by the amount. See Houlgate (2018), 191.
8. In the second edition Hegel undertakes a similar revision of the sections on determinate being and something; see vol. 1, chapter 8, notes 10 and 16, and Houlgate (2006), 320-1.
9. Both Miller and di Giovanni translate "vorzugsweise" as "by choice". See SLM 317.
10. See Wolff, M. (1986), 244: "*x* decreases by as many times [*um sovielmal*], as *y* increases". Note that on LS 357 (l. 28) Hegel states that in the inverse ratio one quantum is "greater *by as much* [*um so viel*] as the other is smaller" (emphasis added). This, however, should be understood, as it is by Michael Wolff, as saying "by as many times" (*um sovielmal*). Di Giovanni's translation overlooks the words "um so viel" and states simply that a quantum is "so connected to an other that the greater it is, the smaller is the other" (SL 276 [ll. 35-6]).

11. In the inverse ratio 3 : 12, if 3 doubles to become 6, 12 must be halved to become 6, too. So, by increasing, 3 deprives 12, not just of the 3 it adds to itself, but rather of 6.
12. See Houlgate (2018), 192-3.
13. See also 2: 163.
14. On SL 190-1 / LS 241-3 the infinity of self-relation is also attained "beyond" the quantum concerned. There, however, such self-relation is not yet explicitly qualitative, but is simply the continuation of one quantum in another *quantum*. See 2: 160-1.
15. For a brief but clear account of the different aspects of the exponent of the inverse ratio, see Pierini (2014), 120.
16. See this chapter, note 1.
17. It is also numerically distinct from each of them, except in the case in which one has the value of 1.
18. See SL 189 / LS 240, and 2: 153-5.
19. Di Giovanni has "self-surpassing".
20. See also Hartmann (1999), 116.
21. Burbidge (2006), 51. See SL 278 / LS 359: "In the ratio of powers it is of a wholly *qualitative* nature".
22. See Wolff, M. (1986), 249.
23. See Houlgate (2014), 28, and Pierini (2014), 120.
24. See SL 176, 240 / LS 223, 308, and EL 162-3 / 215 [§ 102 R]. Note that if one combines Hegel's concept of the power-ratio with those of negation or the direct ratio, then his theory also allows for negative and fractional powers, though once again they would not count as logically necessary in Hegel's sense.
25. See 1: 295-6.
26. See 2: 153-4. This "quality" of the concrete quantum should not, however, be confused with the qualitative something that is shown to be inseparable from the concrete quantum in 1.2.2.B.b.
27. Note that the quantum first becomes qualitative itself – as opposed to just having the "quality" of being self-external, or to just belonging to a qualitative something – when (in 1.2.2.C.c) it becomes the truly infinite quantum that exhibits the *quality* of self-relation – of being for itself – in its relation to another (see 2: 164-5). In the ratio of powers, however, the truly infinite quantum becomes fully and explicitly qualitative.
28. But see also the previous note.
29. See Burbidge (2006), 51. Hegel's remark does not mean, however, that double transitions are found everywhere in the logic of being. We see them in the dialectics of being and nothing, determination and constitution, and extensive and intensive magnitude, but the transitions, for example, from one quantitative ratio to the next are one-way transitions.
30. The quality that is identical with quantity in this case is in turn simple and immediate, or "in a simple determination" (SL 272 / LS 351).
31. See SL 153 / LS 193, and Winfield (2012), 135.
32. See Stace (1955), 169: "measure, therefore, may be defined as [. . .] *quantity upon which quality depends*". The structure of a measure, by the way, does not mean that

a thing's quantum cannot now change into *another* quantum. The defining quality of the thing, however, is tied to the determinacy of *that* quantum (or to a specific range of quanta); so if the latter changes, the quality changes as well and the thing is destroyed (see SL 288-9 / LS 371: "were the latter to exceed or fall short of this quantum, it would perish"). – Note that, in one sense, any quantum possessed by a thing may be said to belong to its measure, for if the thing *has* it, it obviously permits the thing to be what it is (see 2: 248). In that respect, therefore, such a quantum is not a matter of indifference to the thing. Yet insofar as the quantum falls within a *range* of quanta that is the thing's measure, and can be altered without taking the thing beyond that range, it is, indeed, a matter of indifference to the thing in the way we have explained.

Chapter 10

1. Hegel includes a third remark that discusses, among other things, Cavalieri's "method of *indivisibles*" (SL 265 / LS 342).
2. See Boyer (1959), 284 ff.
3. For a shorter treatment of the material discussed in this chapter, see Houlgate (2018), 201-16. See also Moretto (1986), Wolff, M. (1986), and Stekeler-Weithofer (2005).
4. See Wolff, M. (1986), 257.
5. See SL 192, 201-2 / LS 244, 256, and 2: 162.
6. Hegel actually writes "$dx^n = nx^{n-1}dx$". Since, however, $y = x^n$, Hegel's equation is the same as "$\frac{dy}{dx} = nx^{n-1}$".
7. See Thompson (1914), 6: "*dx* is the small increment added by growth".
8. For Frege's somewhat different use of the term "function", see FR 133-4 / FB 5-6, and 2: 68-9.
9. See 2: 231-3, and SL 241-2 / LS 310-11.
10. See 2: 162, and SL 192, 201-2, 206 / LS 244, 256, 262.
11. In the main text Hegel derives the true quantitative infinite, not from the idea of the "infinitely great or small", but from the quantitative infinite progress that is itself generated by the idea of quantitative change. It is only this latter derivation that is properly immanent. See 2: 163-5.
12. See SL 203-4 / LS 258-9, and 2: 165.
13. See SL 278-9 / LS 359-61, and 2: 202 ff.
14. Di Giovanni translates "*an ihm selbst* unendlich" as "infinite *within*".
15. Di Giovanni writes "gone back into the simplicity of self-reference".
16. Recall that the relation of process and *moments* is constitutive of true infinity in the logic of quality. See SL 117-18 / LS 147-9, and 1: 244-5.
17. The other moment, we recall, is being-for-self, *of* which being-for-one is a moment. In becoming a moment itself, however, being-for-self ceases being that and becomes just another being-for-one: one moment in relation to another. Being-for-self is thus not a moment *as pure being-for-self*. See 1: 253-4.
18. "es hat nur Bedeutung in Beziehung auf ein im *Verhältnis* mit ihm Stehendes".
19. See SL 127-8 / LS 160-1, and 1: 252-3.

20. See SL 208 (ll. 13-15) / LS 265 (ll. 21-3).
21. This is misprinted in di Giovanni's translation as "to *dx*".
22. See also SL 219 (l. 34) / LS 281 (ll. 8-9).
23. In the development of the binomial $(x + dx)^n$ dx appears in higher powers of itself, but neither it nor dy appears in this way in $\frac{dy}{dx}$. $\frac{d^2y}{dx^2}$ is, of course, not the square of $\frac{dy}{dx}$ but simply the derivative of that derivative, and so the second derivative of the original function.
24. See SL 211 (ll. 13-18) / LS 269 (ll. 28-35).
25. In this respect, Hegel's conception of $\frac{dy}{dx}$ accords with the modern conception, though, for him, this "single indivisible sign" is not "completely divested of any idea of ratio" (see Boyer [1959], 254; see also 196, 216, 221).
26. See SL 211 (ll. 13-20) / LS 269 (ll. 28-38).
27. See SL 214 (ll. 33-6) / LS 274 (ll. 21-4).
28. See Moretto (1986), 181, and Johnson (1988), 57.
29. Note that irrational numbers do not require consideration here, since they cannot be expressed as fractions. According to Hegel, they have a "higher kind of infinity" that consists precisely in their "incommensurability, or the impossibility of displaying as a quantum, even in the form of a fraction, the quantitative ratio that they contain" (SL 211-12 / LS 270).
30. See Lagrange (1847), 5 [Intro.], and Boyer (1959), 190.
31. See Lagrange (1847), 5 [Intro.]: "dégagé de toute considération d'infiniment petits", 7-8 [I, i, § 1]: "*i* étant une quantité quelconque indéterminée".
32. Lagrange (1847), 1 [Intro.].
33. F (x + i) is Lagrange's general expression for a function of x-plus-an-increment. See Lagrange (1847), 7-8 [1, i, § 1].
34. See also Lagrange (1847), 14-15 [1, i, § 6], and Wolff, M. (1986), 205.
35. See Hartmann (1999), 130.
36. See Wolff, M. (1986), 220-1, and 2: 213.
37. Though $\frac{dy}{dx}$ is, of course, itself a finite quantum (e.g. 2x, $3x^2$ and so on).
38. See Newton (1999), 646-7 [2, ii, Lemma 2].
39. If $y = x^3$, this yields the following values for the two variables: y = 8, x = 2; y = 27, x = 3; y = 64, x = 4; y = 125, x = 5. X increases in each case by 1, but y increases correspondingly by 19, 37, 61. There is, therefore, no constant direct ratio between the increases in the two variables.
40. This derivative relation, Hegel notes, is "*constant*" (*das* Stetige), even though it holds between the qualities, "principles", or respective "hows" of two *variable* magnitudes (SL 218 / LS 279-80). Di Giovanni translates "das Stetige" as "what is *continuous*". – Note that Hegel's conception of $\frac{dy}{dx}$ as the relation between the qualities of two variables, or the ways in which they change with respect to one another, is his alternative to – or, perhaps, his revised version of – the more familiar idea that it denotes the "instantaneous rate of change" of a function.
41. Di Giovanni writes that there was "an intimation" of the concept in the attempts of others to justify the mathematical infinite.
42. On SL 222 di Giovanni translates "in seiner Bestimmtheit" as "accurately".
43. On Hegel's critique of Newton's mechanics, see e.g. Houlgate (2005a), 153-6.
44. Newton (1999), 646-7 [2, ii, Lemma 2]. See also Guicciardini (2002), 313-14.

45. Cohen (1999), 106; see also 316.
46. See Newton (1999), 441-2, 646-7 [1, i, Lemma 11, Schol., 2, ii, Lemma 2].
47. Newton (1999), 441-2.
48. Newton (1999), 442 [1, i, Lemma 11, Schol.], emphases added.
49. Newton (1999), 442 [1, i, Lemma 11, Schol.], and SL 218 / LS 279.
50. See Newton (1999), 442 [1, i, Lemma 11, Schol]: "the limits which the ratios of quantities decreasing without limit are continually approaching".
51. See Newton (1999), 647 [2, ii, Lemma 2], and SL 219 /LS 280.
52. See Newton (1999), 647 [2, ii, Lemma 2]: "do not understand them to be finite particles!"
53. In Hegel's view, Newton's ultimate ratio is not a ratio of ultimate magnitudes, but it is still the limit to which the ratio of diminishing *magnitudes* comes closer than any given difference (see SL 218 / LS 279).
54. Euler (1787), 62-4 [1, iii, §§ 83-4, 86].
55. See Euler (1787), 63 [1, iii, § 84].
56. See Newton (1999), 647 [2, ii, Lemma 2].
57. These reflections on Newton and Euler help to explain why Hegel does not conceive of $\frac{dy}{dx}$ as a limit-ratio. To do so, in his view, is to conceive of it as the ultimate ratio that is approached by *quanta* in relation – namely, increments or decrements to x and y – as they tend to zero. It is, however, "self-evident" "that a quantitative difference which, by definition, not only *can be*, but *ought to be*, smaller than any given [difference], is no longer quantitative" (SL 229 / LS 293). Implicit in the idea of a limit-ratio, therefore, is the thought of a ratio that is no longer quantitative, but rather a qualitative ratio between quantitative moments. Such a qualitative ratio, however, is not itself the *limit of a quantitative one*, but is an altogether different ratio that cannot be reached by reducing quanta to zero (or almost zero). It is a ratio to which the idea of a limit points, but that is not directly captured by that idea (see SL 228-30 / LS 291-4). – Lagrange, on whose work Hegel draws, also drops the idea of limit, though he does not conceive of $\frac{dy}{dx}$ in qualitative terms; see Lagrange (1847), 2-3, 5 [Intro.]: "dégagé de toute considération [. . .] de limites"; Boyer (1959), 251, and Klaucke (1990), 134.
58. The term "derived function" comes from Lagrange; see Lagrange (1847), 2 [Intro.]: "*fonctions dérivées*".
59. It is true that if one includes an additional constant "b" in the original linear function (as Hegel does in this quotation), then the derivative is no longer equal to $\frac{y}{x}$. So if y = ax + b, then $\frac{dy}{dx}$ is a, but a is in turn equal to $\frac{y-b}{x}$, not to $\frac{y}{x}$. As we will see, however, $\frac{dy}{dx}$ corresponds to the slope of the tangent to the line produced by the original function. In the case of a power-function that "line" is a curve, so there is a qualitative, as well as mathematical, difference between it and its tangent. In the case of a linear function, by contrast, the line and its slope or tangent are the *same*, and the additional constant merely changes the point on the y-axis at which the line starts. The addition of a constant to a linear equation does not, therefore, undermine Hegel's argument that there is no particular point in differentiating linear equations.
60. Wolff, M. (1986), 257.
61. Di Giovanni translates "Das Eigentümliche" as "the essential difference" (on SL 238).

62. See SL 176, 240 / LS 223, 308; Wolff, M. (1986), 218-19, and 2: 204.
63. By the way, the fact that Hegel's examples of power-functions are invariably of the form x^n, where n is a natural number (see e.g. SL 252 / LS 326), does not mean that, for him, calculus should not seek the derivatives of more complex power-functions such as $f(x) = e^{-\left(\frac{1}{x}\right)^2}$ (see Cauchy [1823], 152). The qualitative moment of being-for-self is, however, easier to see in simpler power-functions based on squaring and its repetition, that is, in x^2 (or ax^2), x^3, x^4 and so on.
64. Note that there is a mistake on LS 310 and LSGW 280, where the expansion of the function is written as "($y + ny^{n-1}z + \ldots$)", rather than (as it should be) "($y^n + ny^{n-1}z + \ldots$)". This mistake is also found in vol. 5 of *Werke in zwanzig Bänden* (WL 1: 331), and in both the di Giovanni and Miller translations; see SL 241 and SLM 281. The mistake is noted in Stekeler-Weithofer (2005), 255. It was also spotted by a keen-eyed student, Simon Lee, in 2015.
65. See SL 223 (ll. 14-17), 242 (ll. 22-5) / LS 285 (ll. 23-7), 312 (ll. 10-13).
66. See Wolff, M. (1986), 220-1. Lagrange also maintains that derived functions can be generated, not only by expanding a binomial, but also by repetition of the relation between derivative and original. See Lagrange (1847), 19 [1, ii, § 9], and Klaucke (1990), 133. Yet Lagrange conceives of such functions as essentially moments of an ongoing *series* generated by repetition, whereas Hegel sees them simply as *repetitions* of the relation between derivative and original.
67. Di Giovanni translates "Beziehung" as "connection".
68. SL 230, 235, 253, 259 / LS 294, 301, 327, 335. See also Lagrange (1847), 8 [1, i § 1], and Klaucke (1990), 131.
69. Lagrange (1847), 18-19 [1, ii, §§ 8-9]. See also Boyer (1959), 252, and SL 226n / LS 288n.
70. Boyer points out that in the early nineteenth century "doubts began to arise as to the correctness of the principle" – propounded by Lagrange – "that all continuous functions could be expanded in Taylor's series" (Boyer [1959], 261, 267). As Wolff notes, however, such doubts do not appear to interest Hegel (Wolff, M. [1986], 220-1). The reason why, as Wolff indicates, is that, for Hegel, the object of calculus is the relation between a derivative and the original power-function (and between any subsequent derivative and its "original"); it is not (as it is for Lagrange) the *series* or *sum* of such derivatives. The form of the series is, in Hegel's view, a useful *means* for discovering the derivatives of a function, and it works unproblematically in the case of simple power-functions, such as $y = x^3$. It is, however, no more than a means, since the derivatives are not to be understood essentially *as* members of a series. If, therefore, there are cases where a differentiable function cannot be expressed, by means of Taylor's theorem, as a series, this does not undermine Hegel's contention that a derivative expresses a qualitative relation that can be derived from a power-function. – By the way, Hegel's claim that the proper role of differentiation is to discover the derivatives of *power-functions* suggests that he would not be concerned by the proofs, provided by Bolzano and Weierstrass, that there are continuous functions that are non-differentiable (see Boyer [1959], 269-70, 284-5). These proofs would concern him if his claim were simply that all *continuous* functions have derivatives. As we have seen, however, this is not his claim; indeed, he states explicitly that "continuity"

is too abstract and empty a category to ground the possibility of differentiation, as philosophy conceives it (SL 230n, 239 / LS 294-5n, 307, and Wolff [1986], 254-5).

71. See also SL 254 / LS 329, and Wolff, M. (1986), 225, 229, 251.
72. Di Giovanni's translation obscures Hegel's point by omitting the equation at issue. It says simply: "All that the equation $\frac{dy}{dx}$ expresses . . .". Miller reproduces the text correctly (see SLM 291).
73. Functions with higher powers, such as x^3, x^4, will also have a tangent, subtangent and so on, so Hegel's remarks are not restricted to quadratic functions.
74. For definitions of the ordinate and abscissa as magnitudes, rather than lines, see 2: 223.
75. See Wolff, M. (1986), 229-30.
76. For ease of access I have taken the diagram from https://en.wikipedia.org/wiki/Subtangent. It should be noted, however, that the curve in the diagram is not simply "determined by an equation of the second power".
77. See Thompson (1914), 77-8.
78. Silvanus P. Thompson argues that, if $\frac{dy}{dx}$ is the slope of the tangent to the curve at a defined point, then $\frac{d^2y}{dx^2}$ indicates whether the slope becomes greater or less as the value of x increases. If $\frac{d^2y}{dx^2}$ is positive, the slope is becoming greater, but if it is negative the slope is diminishing. In Hegelian terms, therefore, the second derivative indicates *how* the slope – which is a constant function, such as 2x or $4x^3$ – changes its value with changes in the value of x. See Thompson (1914), 112-13.
79. See Stekeler-Weithofer (2005), 256. Note, by the way, that Hegel's conception of $\frac{dy}{dx}$ still leaves space for the familiar idea that calculus can determine the ratio between the increase of one magnitude and the increase of another. In Hegel's view, $\frac{dy}{dx}$ in its "concept" is a relation between qualitative moments, not "increments". Nonetheless, the example of the ordinate and subtangent shows that $\frac{dy}{dx}$ can *match* the ratio between magnitudes, or between their increments and/or decrements, and so determine its value.
80. See SL 250 (ll. 5-7) / LS 322 (ll. 25-8).
81. Boyer credits Karl Weierstrass with "effectively banishing from the calculus the persistent notion of the fixed infinitesimal" (Boyer [1959], 287), but Hegel banished this notion from calculus several decades before Weierstrass. Like Hegel, Weierstrass also banishes the idea that a function, f(x + dx) – fx, "approaches" its limit, $\frac{dy}{dx}$, as dx tends to zero (see Boyer [1959], 286-7, and SL 231 / LS 297: "dx has already left approximation behind; it is neither near, nor nearer"). Unlike Hegel, however, Weierstrass continues to regard $\frac{dy}{dx}$ as a *limit* – albeit one that "does not involve the idea of *approaching*, but only a static state of affairs" (Boyer [1959], 287). For Weierstrass (in Boyer's words), "the number *L* is the limit of the function *f(x)* for $x = x_o$ if, given any arbitrarily small number ε, another number δ can be found such that for all values of *x* differing from x_o by less than δ, the value of *f(x)* will differ from that of *L* by less than ε" (Boyer [1959], 287; see also Wolff, M. [1986], 209 [n. 27]). Weierstrass thus conceives of a function's limit – and of the derivative – in terms of the possibility of a "*less than*". In contrast to Hegel, therefore, he continues to conceive of the derivative quantitatively, rather than qualitatively. – Hegel, of course, knew nothing of Weierstrass's work. In the early nineteenth

century, however, Cauchy put forward a conception of limit that anticipates that of Weierstrass, although it still involves the idea of "approaching". Cauchy states that "when the values successively attributed to the same variable indefinitely approach a fixed value, in such a way that they end by differing from it by as little as one likes [*aussi peu que l'on voudra*], this latter is called the *limit* of all the others" (Cauchy [1823], 1). Wolff contends that Hegel has Cauchy's conception in mind when he characterizes the limit (dismissively) as "that determinate value which the ratio can get *infinitely near to*, that is, so near that the *difference* can become *smaller than any given one*" (SL 229 / LS 293, and Wolff, M. [1986], 232-3). Andreas Klaucke, by contrast, thinks that Hegel is more likely to be referring to the work of Sylvestre François Lacroix (Klaucke [1990], 143-4). Either way, what is striking is the parallel – leaving aside the idea of approaching or getting near to – between Hegel's association of the limit with the possibility of a difference being "*smaller than* any given one" and Weierstrass's association of the limit with the possibility that a function can differ from it "by *less than* ε". This parallel makes it clear that, despite the years that separate them, Hegel would not consider Weierstrass to have come any closer than his contemporaries to the philosophical concept of the derivative, which, for Hegel, is implicit in but distinct from that of the limit as such (see this chapter, note 57). – Note that in this chapter I have focused solely on Hegel's conception of differential calculus. For his account of integration, see SL 253-8 / LS 327-34.

Chapter 11

1. A shorter version of this chapter can be found in Houlgate (2017b).
2. See vol. 1, chapters 13 and 14, and Houlgate (2014), 17-19.
3. See Moretto (2000), 33-6. On the relation between Hegel's concept of measure and Spinoza's conception of "mode", see Ferrini (1988).
4. See SL 293-4, 296-8, 303-5 / LS 377-8, 381-3, 390-2.
5. See SL 9 / LS 6; Houlgate (2006), 115-43, and vol. 1, chapter 5.
6. See Houlgate (2005a), 115-21. By contrast, Ulrich Ruschig contends that, in the chapter on real measure at least, the logical transitions are determined by the scientific models that Hegel cites as mere "examples"; see Ruschig (1997), 16, 28, 233, and Ruschig (2000b), 304-5. Günter Kruck argues, as I do here, that Hegel's account of measure proceeds *logically* and is intelligible "without the integrated, concrete material of intuition" supplied by science; see Kruck (2014), 123-4.
7. See SL 153 / LS 193, and 1: 294.
8. Di Giovanni translates "ihr zugehörigen" as "appropriate to it". See also Moretto (2002), 76, and Winfield (2012), 145.
9. See also EL 171-2 / 226-7 [§ 108 A], and Stace (1955), 169-70. Ruschig points out that water starts to evaporate below 100° C, but that at boiling point water below the surface also turns to steam (Ruschig [1997], 287).
10. See SL 289 / LS 372-3: "The quantitative determinacy is thus twofold [. . .]". – Water preserves its chemical formula, H_2O, whether it is in a liquid or gaseous state, so one might think that the change from one state to the other does not take it

beyond its qualitative limit. In such change, however, a body of liquid water does not preserve its identity *as* liquid water, and so does not merely *change* (in the manner described in 1.1.2.B.a), but it becomes something with a different defining quality, namely steam. In that sense, the change of state takes water (as liquid) beyond its qualitative limit.

11. See this volume, chapter 9, note 32.
12. See also SL 320, 322 / LS 412-13, 415, and 2: 325-6.
13. On the early history of these paradoxes, see Moline (1969).
14. See Ferrarin (2001), 136. Note that someone can be described as bald *before* he (or she) has lost every hair, so the paradox cannot be resolved simply by saying that only removing the last hair makes one bald. The paradox arises because removing hairs one by one initially does not make someone bald, but then does do so, even though one has simply continued to remove one hair at a time.
15. From the Hegelian point of view, the phenomenon exposed by sorites paradoxes is thus not principally the "vagueness" of concepts such as "heap" or "bald", but (in Errol Harris's words) "the interdependence of the moments of measure" – the fact that at some point a quantitative change makes a qualitative difference (Harris [1983], 147). For a good account of Hegel's understanding of the paradoxes (and of "gradualness"), see Pechmann (1980), 90-6, and for a discussion of the relation between the paradoxes and vagueness, see Hyde 2011.
16. See Houlgate (2006), 312-27, and vol. 1, chapter 8.
17. Hegel employs the phrase "the mere quantum" (*das bloße Quantum*) on SL 291 / LS 375.
18. Di Giovanni has "diverse concrete existence".
19. See SL 285-6 / LS 368: "The development of measure contains the differentiation of these moments".
20. The thought of a rule or standard is introduced by Hegel in the first sub-division of the first chapter on measure (1.3.1.A) – in which he examines the initial immediate measure, or "specific quantum" – but, strictly speaking, that thought does not belong there (see SL 289 / LS 371-2). A rule or standard is a measure that, unlike the "specific quantum", is explicitly *distinct* from the quantum to which it relates, and so it belongs in the *second* sub-division of that chapter (1.3.1.B.a). (It is introduced in the first sub-division only to exemplify measure "in the usual sense".)
21. See Houlgate (2014), 20-2, and 2: 11.
22. Measure is, at the start of its logical development, "its own determinacy [*Bestimmtheit*] in itself" (SL 288 / LS 371), and such determinateness remains a feature of measure throughout that development.
23. See Burbidge (1996), 27-30, and Burbidge (2006), 53-4.
24. Burbidge (2006), 54.
25. See SL 291-2 / LS 375.
26. See SL 96-7 / LS 119-121; Houlgate (2006), 348-56, and 1: 194-5.
27. Note that in the first edition of the *Logic*'s "doctrine of being", Hegel continues to call this measure a "rule"; see WLS 230: "In the *rule*, on the contrary [. . .]" (*In der* Regel *hingegen* [. . .]). In the second edition this is changed to: "In the specifying measure, on the contrary [. . .]" (*In dem spezifizierenden Maße hingegen* [. . .]) (SL 292 / LS 376).

28. See Pluder (2004), 59: "the specific quantum alters the alteration, turns the alien alteration into its alteration, and precisely in this specifying shows itself to be specific".
29. Di Giovanni translates "Potenzbestimmung" as "exponent" (and "exponential determination"). – The first, immediate, form of measure was not a power-determination because it was, precisely, an *immediate* quantum. The specifying measure, by contrast, is explicitly distinguished, as an actively negating *measure*, from the quantum, and so can no longer just be a mere quantum itself, but must be one that is explicitly qualitative, namely a power-determination. The rule is also explicitly distinguished from the mere quantum, but since it is principally just something *other* than that quantum, it is an immediate quantum itself (albeit one that provides a standard for its counterpart).
30. See WLS 233, where the relation of a specifying measure to a quantum is described as a "qualifying, that is, raising to a power [*Potenzieren*] of the external quantum". – Alexander von Pechmann provides a clear and illuminating discussion of the specifying measure and, in an interesting move that goes beyond Hegel's own account, highlights the similarities and differences between the specifying measure and the concept of a "function" (see Pechmann [1980], 115-33).
31. See Calle (2001), 206-7.
32. See Doz (1970), 118.
33. See Johnson (1988), 71.
34. Accordingly, in the first edition Hegel claims that if, on a graph, one plots changes in the temperature of a body against changes in external temperature, this will produce a curve ("eine krumme Linie") (WLS 233). See also Moretto (2002), 84.
35. See PN 2: 304.
36. Rogers (2005), 1.
37. Vallance Group, 3, emphasis added.
38. See White and Collocot (1984), 1251-2, 1255. Thanks are due to Thomas Shaw for this reference.
39. Hegel's remarks indicate, however, that, in his view, early nineteenth-century science had already shown specific heat to vary with temperature. I should like to thank Thomas Posch, who sadly died in 2019, for the help he provided with this section on specific heat. For my obituary for Posch, which includes a brief account of his excellent book on Hegel's theory of heat (Posch [2005]), see Houlgate (2020).
40. See SL 203, 279 / LS 258, 360: "The *externality* of the determinacy is the quality of quantum", and WLS 234: "quantity is itself a quality over against quality as such". See also 2: 205.
41. See EL 141 / 188 [§ 88 R].
42. In Hegel's example of specific heat, the body concerned will (usually) have a different defining quality from the surrounding air. The two must, however, both be capable of absorbing heat, if the temperature of the air is to affect the temperature of the body. In this respect, therefore, they must both have the "same quality".
43. This is not to deny that the two quanta may be the same, but in each case it is the specific quantum of *its* quality.
44. Di Giovanni has "each a qualitative existence for itself". See also SL 298 / LS 384.

45. Each, therefore, has the form of "something" (as in 1.1.2.B.a.1) before the idea of "being-for-other", or other-relatedness, is introduced.
46. See also SL 297 / LS 382: "their quantitative relation, the *being-for-self* of measure, is only *one* measure-determination".
47. For the term "*Maßbestimmung*", see SL 294 / LS 379.
48. Note that space (or distance) and time are regarded here as *qualities* whose quanta stand in a certain relation (see SL 295 / LS 380). In the philosophy of nature, however, the "quality" of space is itself understood to be "pure *quantity*" (see EN 29 / 42 [§ 254 R]).
49. Hegel describes simple uniform velocity as a "formal" determination that does not actually exist, but that belongs to "abstracting reflection" (SL 296 / LS 381). This is because in the real world velocity can change in more or less subtle ways as a body moves through space. Hegel is not, however, denying that one can travel for periods of time at a more or less constant speed, or that a body can cover a certain distance at a definite average speed.
50. See Houlgate (2005a), 138-44.
51. See Houlgate (2005a), 147-53.
52. See Pechmann [1980], 136-40 on the realized measure and the three forms of motion associated with it. – It is evident from Hegel's examination of the realized measure that his interest in Galileo and Kepler arises not just from their importance for science but from the fact that their laws express *necessary measures* (see SL 297-8 / LS 383). On the relative significance of Kepler and Newton, in Hegel's view, see Houlgate (2005a), 155-6.
53. See WLS 236: "The qualities are thus essentially distinguished according to the determinate character of the quantitative moment of measure".
54. Hegel notes, however, that in simple velocity it is ultimately a matter of indifference which is the amount and which the unit. So one can also understand velocity in terms of the time it takes to traverse a unit of distance (SL 296 / LS 382).
55. Hegel's more detailed explanation is to be found in his philosophy of nature; see EN 59 / 78 [§ 267 R], and Houlgate (2005a), 140-2.
56. Note that although $s = at^2$, time is to be regarded as the "root" insofar as it is simply t.
57. See EN 71 / 92-3 [§ 270 R], and Houlgate (2005a), 151-3. Hegel's argument, very briefly, is as follows. Time, as the negativity inherent in space, is self-relating negation, and insofar as it relates *to itself* in this way, it must appear in this Keplerian measure as multiplied by itself, that is, as *squared*. Space, by contrast, extends itself in three different directions and so must appear in the measure as a *cube*. Hence $s^3 = at^2$.
58. This is evident in the *Philosophy of Nature*, too, since Hegel there describes space as "pure *quantity*" (EN 29 / 42 [§ 254 R]). It is ultimately quantity, therefore, that sets its qualitative expressions in a determinate quantitative relation to one another.
59. See Moretto (2000), 48-50.
60. On the relation between Hegel's speculative philosophy and empirical science, see Houlgate (2005a), 115-21. For an alternative interpretation of Hegel's logic of measure that rejects the idea that Hegel provides "an 'a priori' grounding for the laws of Galileo, of Kepler", see Stekeler-Weithofer (2018), 247-8.
61. In speculative logic Galileo's law of fall and Kepler's third law of planetary motion are just examples, but in his philosophy of nature Hegel argues that these laws are

themselves made necessary by the logic of nature (that is, of space and time). See SL 297 / LS 382; EN 58-9, 71 / 77-8, 92-3 [§ 267 R, § 270 R], and Houlgate (2005a), 138-153. To my mind, therefore, John McCumber is mistaken in his claim that "nature for Hegel is philosophically, and so ultimately, not a law-governed domain but a 'system of stages'" (McCumber [2014], 65). Nature, for Hegel, is indeed a system of stages (EN 20 / 31 [§ 249]), but that does not prevent it from giving itself its own laws and in that sense being *auto-nomous* (or prevent being as such from giving rise to the measures exemplified by those laws).

62. For Karin de Boer, by contrast, Hegel's logic as a whole is such a reconstruction, and for Robert Pippin the doctrine of being as a whole is such a critique. See de Boer (2010), 39-41, and Pippin (1989), 191-201.
63. Note that the *categories* of "contingency" and "necessity" are not derived in the *Logic* until the doctrine of essence (see SL 478 ff. / LW 176 ff.). Before that point, however, we can describe some moments as contingent and some as necessary, and indeed we can describe the contingency of some moments itself as necessary – just as we describe "being" and "nothing" as *other* than one another before the category of the "other" has been derived. On the difference between the categories thematized in the *Logic* and those employed in thematizing categories, see Houlgate (2006), 83-8.
64. On the limits of philosophy with respect to nature, see EN 23-4, 62 / 35-6, 82 [§ 250 R, 268 A], and Houlgate (2005a), 112-15.
65. See EN 33-4 / 47-8 [§ 257 and A], and Houlgate (2005a), 127-9.
66. See Pechmann (1980), 143-4. Di Giovanni's translation omits the phrase "according to the determinacy of magnitude" (*nach der Größebestimmtheit*).
67. See 2: 265-6, and this chapter, note 57.
68. See WLS 240: "the ratio of powers has some quantum as its basis, which relates to itself in it".
69. Note that Hegel's "a" is simply an empirical constant. In the more usual expression of the law of fall this constant is $\frac{1}{2}$a or $\frac{1}{2}$g, where a or g represents the constant rate of acceleration of the falling body. The law is thus expressed as $s = \frac{1}{2}at^2$ or $s = \frac{1}{2}gt^2$. See Calle (2001), 37-40, and Houlgate (2005a), 139.
70. Strictly speaking, this empirical coefficient is being-for-self only *implicitly* (*an sich*), since it is simply a fixed immediate quantum and does not exhibit the developed character of being-for-self we encountered in 1.1.3.
71. Hegel notes that the first temporal moment in a fall is "only an assumed unit" (say, a second) (SL 299-300 / LS 386). Nonetheless, one can calculate the distance a body must fall in such a unit of time, and the result is the *direct* ratio s = at. "*a*" is thus the exponent of a definite direct ratio that belongs inseparably to constantly accelerating motion, but that determines an initial uniform motion that is itself merely "*imagined*". (On uniform velocity, see also this chapter, note 49.)
72. See Doz (1970), 141, and Ferrini (1998), 300-1.
73. Hegel himself states that science in the early nineteenth century has so far not explained what grounds such numbers (SL 300 / LS 386).
74. See EN 60 / 79 [§ 267 A], and Houlgate (2005a), 142.
75. The surface gravity on the Moon, for example, is 0.16 of that on Earth, and on Mars it is 0.38. See Sparrow (2006), 72, 88.

76. The mean distance of Mars from the sun is 227.9 million kilometres and its orbital period is 687 earth days. If one divides the cube of the former (omitting "million") by the square of the latter, the result is 25.079. The mean distance of Jupiter from the sun is 778.3 million kilometres and its orbital period is 11.86 earth years or *c*. 4329 earth days. In this case, the cube of the former (omitting "million") divided by the square of the latter yields the result, 25.158. See Sparrow (2006), 88, 140.
77. Both Miller and di Giovanni mistranslate "als quantitativ" as "as qualitative" (see SLM 346).
78. See this chapter, note 76.
79. In the Galilean measure, the exponent of the measure itself – that is, of $s = at^2$ – is the same as the exponent of the initial direct ratio between the *immediately given* quanta it contains, namely s and t. The two exponents coincide for the following reason. If a body falls 5 metres in 1 second, the exponent of the initial ratio between distance (s) and time (t) is 5. To calculate the distance the body falls in 2 or 3 seconds, we then simply multiply the initial distance by t^2: the new distance is thus 5×2^2 or 5×3^2 metres. Clearly, if we then *divide* the new distance by t^2 to find the exponent of the ratio between them (namely *a*), we will once again get 5. The ratio between s and t^2 thus has the same exponent as the initial ratio between s and t.

Chapter 12

1. The specifying measure (in 1.3.1.B.b) is also a measure in relation and also involves a power-determination. The relation, however, is not explicitly between quanta of certain *qualities* (such as space and time), but is simply between two quanta, one of which is raised to a power and one of which is not.
2. In this second case there will also be a direct ratio between s and t, but it will not have as its exponent the *a* in $s^3 = at^2$. See 2: 274.
3. See vol. 2, chapter 11, note 76.
4. This, of course, does not apply to the first form of realized measure, simple velocity, since it is just a direct ratio between the quanta of two qualities.
5. See Doz (1970), 142.
6. Di Giovanni translates "Insichsein" as "being-in-itself", which is also the term he uses (correctly) to translate "Ansichsein" (see SL 92 / LS 114).
7. In this case di Giovanni translates "Insichsein" as "in-itselfness".
8. Di Giovanni translates "materielle Teile" simply as "parts". On weight, see also EN 49 / 66 [§ 265]. – An object with mass is, of course, infinitely divisible insofar as it occupies space, and so in that sense it does not contain a definite *amount* of parts. Yet its mass is itself a definite amount that can be determined by measuring its weight, and so in that sense the object can be said to contain a definite amount of parts after all.
9. The density of a material is its mass per unit of volume, and its specific weight is its weight per unit of volume. The density is fixed, but the specific weight is the density in relation to a gravitational field and varies with the latter. Specific gravity is the relation between the density of one substance and that of another reference substance, such as water. On specific weight, see Ruschig (1997), 27.
10. See EN 128 / 160 [§ 293 A].

11. See Ruschig (1997), 36, and Schick (2014), 140, 142. Things may, of course, share a specific weight and the latter may thus fail to specify a thing properly (see Ruschig [1997], 44). At this point in the *Logic*, however, the distinguishing quality of each side of a real measure consists in nothing more than its specific weight or density (or, rather, is exemplified by the latter). This, by the way, is not to deny that density is in turn a ratio between *qualities* of a thing (namely, its mass and volume). Density as such, however, is only the quantitative *ratio* between such qualities.
12. See SL 98-9, 170 / LS 122-3, 215.
13. See also SL 306 (ll. 10-11) / LS 393 (ll. 16-18).
14. Density is the ratio of mass to a unit of volume, and so the volume counts as 1; the exponent of the density is thus the quantum of its mass (or weight): say, 4. If that density is then set in relation to another density, whose exponent is 1 (such as water), then the exponent of that very relation will be the same as the exponent of the first density: so once again 4. If, however, the first density is set in relation to another density, whose exponent is greater or less than 1 – say, 2 – but nonetheless counts as the unit in the relation, then the exponent of that relation will be the exponent of the first density divided by that of the second: so in this case 2.
15. Strictly speaking, specific gravity is density *in* relation to another; but it can also be conceived *as* that relation and in that sense coincides with the real measure itself.
16. As a matter of physical fact, density – and thus specific gravity – can vary with temperature and pressure, since changes to the latter can increase or decrease volume without changing mass (see Burbidge [1996], 31, and Ruschig [1997], 35). If, however, temperature and pressure are constant, so, too, is the density. Moreover, temperature and pressure are not part of the logical structure of density itself; so from a purely *logical* point of view density (and the measure it exemplifies) is also fixed and unchanging.
17. Unless physical reasons prevent this, as, for example, with the speed of light.
18. See this chapter, note 16: in nature density can be affected by temperature and pressure, but according to the pure *logic* of real measure all that is available here to change a density is another density.
19. See Doz (1970), 145: "the quantum is *conceptually* variable", but "here a given measure will be changed in reality [*réellement*] by its contact with a different measure". See also Schick (2014), 143.
20. Di Giovanni translates "Verbindung" as "composition". – Ruschig points out that the density of a metal can also be changed by simply melting the metal concerned (Ruschig [1997], 47). This, however, is irrelevant here, since such a change, though a physical possibility, is not grounded in the *logical* structure that has emerged at this point in the *Logic* – namely, the relation of one measure-as-ratio to another measure-as-ratio.
21. See SL 304 / LS 391; EN 236-8 / 292-4 [§ 327 and A]. This is not to deny that metals can react chemically with water, but such chemical reactions are different from the simple combination of densities that Hegel has in mind here.
22. By contrast, when two simple quanta are added together, what results is just the sum of the two, without any reciprocal "specification".
23. See EN 237 / 293 [§ 327 A]: "water and alcohol mixed together [. . .] occupy a smaller space", and Ihde (1984), 98: "he [Berthollet] pointed out that the volume

of ammonia is significantly less than that of the elemental gases of which it is composed – in fact, one-half".

24. See Ruschig (1997), 16, 46, 57, and Kruck (2014), 124n.
25. For Hegel's account of the way in which the many ones form one "one of attraction", see SL 139, 141 / LS 175-6, 178-9, and 1: 275-6.
26. See Pechmann (1980), 177-8; Pluder (2004), 97-8, and Schick (2014), 144. Burbidge provides a helpful account of this aspect of real measure, but he grounds the requirement that a measure be combined with other measures in *our* need to achieve "a more accurate measurement" of the thing, rather than what is needed for the thing to distinguish *itself* qualitatively from other things (see Burbidge [1996], 32-3).
27. Note that di Giovanni distorts Hegel's sense in the lines quoted by translating "Das Verhältnis solcher Reihe innerhalb ihrer" as "the relation of such series among themselves" (rather than "the relation of such *a* series within *itself*"). Miller gets Hegel's sense right by writing: "the interrelationship of the members of such a series" (SLM 352).
28. Di Giovanni translates "unbestimmt andere" as "do not differ in any determinate way". – Pechmann maintains that, while it is logically necessary for one measure, X, to combine with a series of other measures, A, B, C – in order to display its distinctive character – it is not logically necessary for X to be compared with Y (Pechmann [1980], 178-80). Measures as quanta, however, are necessarily one of many and so can always be compared with other measures *beyond* the specific series with which they combine. There is, therefore, a logical, not just an "extra-logical", reason why Hegel now considers the relation between two measures, X and Y, that are "indeterminate others" to one another.
29. See SL 325 / LS 419: "the same constant order" (*dieselbe konstante Ordnung*). See also Pluder (2004), 99-103.
30. Emphasis added to "to both" in the quotation. Note that di Giovanni misses out this phrase – "zu beiden" in German – from his translation of this remark.
31. See Doz (1970), 147, and Burbidge (1996), 35-6.
32. The clause "those measures, which, together with the two or rather indefinitely *many* that stand over against them and that are compared with one another" reads in German as follows: "diejenigen [. . .], welche mit den gegenüberstehenden unter sich verglichenen Beiden oder vielmehr *Vielen*". In these lines Hegel refers first (with the words "those measures") to one series of measures (A, B, C) and then to the "many" measures (X, Y, Z) that stand over against that first series and in so doing are compared with one another. Di Giovanni loses sight of this twofold relation of the "many" to one another and to the first series of measures by describing the many as "the indefinitely many *reciprocally* opposing and contrasting measures" (emphasis added). – Note that in the subsequent lines, the measures that are "first named" (and then referred to as "members of the first series") are X, Y, Z, not A, B, C.
33. Note that here, and in the following pages, XA, XB and so on represent the (modified or "specified") *sum*, not the product, of X and A, X and B and so on.
34. Hegel distinguishes between the series of exponents and the series (or Series) of the numbers that produce the exponents on SL 306 / LS 393: "Now inasmuch as this independent measure produces a series of exponents with a series of such independent measures".

35. For a clear and helpful account of the way these two Series of measures "intersect", see Burbidge (1996), 36-7.
36. Di Giovanni translates "Verhältniszahl" as "numerical ratio".
37. Ruschig claims that Hegel has such chemicals in mind from the start of 1.3.2.A.b, but this seems to me mistaken (see Ruschig [1997], 44). In my view, chemicals such as acids and alkalis, rather than simple densities, do not become the appropriate examples for the logical structures concerned until 1.3.2.A.c. This is not to deny that two metals with different densities bond "chemically" (*chemisch*) when they form an alloy; in 1.3.2.A.b, however, Hegel is principally concerned with the combination (and reciprocal "specification") of densities as such. See SL 318 / LS 409-10: "When bodies bond chemically, even if only as amalgams or synsomates, there is likewise evidence of a *neutralization* of specific gravities".
38. See SL 325 / LS 418, where Hegel includes "the combinations of materials of different specific gravities" among the "exclusive measure-relations".
39. Hegel maintains that sounds and musical notes also have such elective affinities for one another (SL 308-9 / LS 396-7). See also Johnson (1988), 81, and Burbidge (1996), 41.
40. Burbidge (1996), 165-6; see also 41, and Burbidge (2006), 56.
41. See Holmes (1962), 106. Certain metals can also displace other metals from aqueous solutions of their salts, depending on their relative positions in the reactivity series of metals. So, for example, magnesium would displace copper from copper sulphate solution (see Online Labs [2011]). Goethe famously drew on displacement reactions in his novel *Die Wahlverwandtschaften* (1809).
42. On SL 310 di Giovanni translates "Vorzug" as "advantage".
43. See 2: 281, and SL 304 / LS 390: "These qualities are quantitatively determined, and their relation to one another constitutes the qualitative nature of the material something – the ratio of weight to volume".
44. See 2: 285-6, and SL 318 (ll. 22-4) / LS 410 (ll. 7-10).
45. See SL 318 / LS 410: "The exponents of these ratios are not exclusive determinations of measure; their progression is continuous".
46. Densities vary with temperature and pressure, but are fixed under fixed conditions; see this chapter, note 16.
47. See SL 317-18 / LS 409, where Hegel states explicitly that chemical affinity – the fact that a chemical combines with more of this and less of that – is different from the relation between mere densities. (On the difference between chemical affinity and elective affinity, see SL 315 / LS 405.)
48. See also Ihde (1984), 96.
49. See Ihde (1984), 97.
50. Elective affinity tables were published in 1778 by the Swedish chemist Torbern Bergman. See Holmes (1962), 106; Ihde (1984), 94, and Burbidge (1996), 166.
51. Ruschig (1997), 119; see also 123, 191. On p. 119 Ruschig actually refers to the relation between stoichiometric relations and the "energetic magnitudes" that determine reactions. A few pages later, however, he notes that elective affinities between substances are today understood precisely in terms of the "energy content of a combination" (126).
52. Ruschig (1997), 132; see also 167. If concentrated sulphuric acid is added to anhydrous sodium chloride, then hydrogen chloride gas is released. If, however,

diluted sulphuric acid is added to aqueous sodium chloride, then the reaction produces hydrochloric acid. – Note that Ruschig recognizes that, for Hegel too, the specifically exclusive quality of elective affinities cannot be derived directly from stoichiometric ratios; but he thinks – mistakenly, in my view – that Hegel reaches this conclusion precisely by trying to derive the former from the latter (see 130-3, 173-4, and Ruschig [2000b], 306-7).

53. SL 309 (l. 22), 311 (l. 5) / LS 397 (l. 35), 399 (l. 36).
54. See Ruschig (1997), 119, 123, 132.
55. Ruschig (1997), 119.
56. SL 311 (l. 5), 312 (l. 4) / LS 399 (l. 36), 401 (l. 5).
57. Of course any musical notes can be played together, but only certain combinations produce harmonies.
58. See Burbidge (1996), 42: "The preference for one combination over others is a function of something more than just quantitative magnitude. It reflects the particular components that make up the compound".
59. See this chapter, note 52.
60. Exponents formed by combining densities are, albeit incompletely, exclusive (and so in that sense qualitative), but the densities themselves are mere quantitative ratios. When the exponents are understood to be fully and explicitly exclusive (and qualitative), however, then the measures that combine to form those exponents must themselves be qualitative, as well as quantitative, and so are exemplified by chemicals rather than mere densities. It is only such chemicals, not mere densities, that can exhibit an elective affinity for one another (see 2: 300-1).
61. Logically, such a quantity could be very small so that the elective affinity shows itself immediately; in other cases, a bigger quantity could be required for the elective affinity to manifest itself. This is for science to establish in any given case.
62. Di Giovanni translates "Vorstellung" as "concept".
63. See also Doz (1970), 152-3, and Holmes (1962), 110-11.
64. See also Holmes (1962), 115: "Berthollet did not deny, as some chemists later claimed, the action of affinities in chemical reactions. He merely denied the possibility of measuring them in the way Bergman had done and called attention to other factors that also exert an action".
65. Burbidge (1996), 68, emphasis added. See also 69, 167.
66. See e.g. EN 262 / 324 [§ 333 R].
67. Ruschig is thus correct to maintain that "there is no elective affinity as pure elective affinity, independent of the conditions that are constitutive for it" (Ruschig [1997], 121). What he fails to see, however, is that such absence of "purity" is made necessary by the *logic* of measure and does not become visible only by presupposing the findings of natural science.

Chapter 13

1. Real measure is from the start a single measure, since it is an "independent whole" (SL 301 / LS 388). Yet it is initially a single *relation between* two direct ratios and in that sense is not explicitly a *single* measure.

2. The moment of exclusion, which belongs to being a *one* (*Eins*), is not initially a feature of measure, since the latter is at first merely an immediate quantum to which an immediate quality is attached (SL 288 / LS 371). The two sides of a real measure, though independent, are not yet exclusive either, since they are moments of a single measure. The combinations they form, however, are exclusive, since they are not – or, at least, not necessarily – moments of a further unity and so enjoy an unqualified independence. See 2: 298-9, and also 2: 300-1 (on the difference between incompletely and fully explicit exclusive unities).
3. On such conditions, see 2: 313-14.
4. See Burbidge (1996), 45: "each of the elements has its own independent life, amenable to entering into all kinds of other compounds". Note that while this is true of the logical structure of the exclusive measure under consideration here, whether and when it is true in nature would have to be determined by the philosophy of nature or by natural science. Musical notes that combine into a specific harmony are most obviously separable, but some chemical reactions are also reversible.
5. On the "stoichiometric" ratios, in which, for example, acids and bases saturate one another, see 2: 305-6.
6. It has to be conceded that Hegel does not make it clear in the first two paragraphs of 1.3.2.B that a new form of measure is emerging. It is evident from the following paragraphs, however, that one has arisen, since the focus there is on the way two measures can form several different combinations with one another (in a nodal line), rather than (as in 1.3.2.A.c) a single combination that excludes combinations with other measures.
7. See SL 320 (ll. 6-7) / LS 412 (ll. 13-14).
8. See Burbidge (1996), 45: "The neutral combination [. . .] is qualitative, a foundation that persists".
9. Hegel writes that the exclusive elective affinity continues itself in other "neutralities" because "the basic determination is quantitative" (SL 319 / LS 410). We can now see that it is more accurate to say that it changes into another unity due to being quantitative, but that it *continues itself* therein because the new unity is formed by the same qualitative constituents.
10. In the first case, we can also say that the quantum specifies which quality will emerge: so a temperature between 1° C and 99° C specifies that water will be a liquid, but that of 100° C specifies that it will be steam. For this reason the first measure is called a "specific quantum" (SL 288 / LS 371). See also SL 319-20 / LS 412: "the quantum shows itself to be specifying". See 2: 246-9.
11. See SL 319 (ll. 38-9) / LS 412 (ll. 2-4).
12. See Burbidge (1996), 46.
13. Burbidge (1996), 165. See 2: 301.
14. See Moretto (2002), 88, and Carlson (2007), 231-2.
15. See vol. 2, chapter 12, note 41.
16. See Carlson (2007), 233: "The nodal line is like a knotted string [. . .]". For a clear and helpful account of the nodal line, see also Pechmann (1980), 185 ff.
17. See Burbidge (1996), 46: "Within a range [. . .]".
18. Note that the nodal line is potentially infinite or endless according to the *logic* of measure, but such lines may not always be infinite in nature, since what Hegel calls

the "impotence" of the latter can prevent it from realizing logical forms exactly. See EN 23-4 / 34-5 [§ 250 and R], and Houlgate (2005a), 113. (Martin, by contrast, contends that the nodal line is in principle finite, not endless; see Martin [2012], 130-1, 136-7.)

19. A qualitative something can, of course, change without becoming a different thing, as we saw in volume 1 (1: 181-3). The qualitative change that is at issue in the nodal line, however, involves one thing, or measure, going beyond its limit and becoming an altogether different thing or measure.
20. See Burbidge (1996), 46-7.
21. Di Giovanni puts this in Latin, even though Hegel puts it in German.
22. Moretto points out that Hegel's demonstration of the necessity of discontinuities in the world shows the limited validity of any "general postulate of continuity in nature", as put forward, for example, by Leibniz (see Moretto [2002], 90, and Leibniz [1998], 269 [*Monadology* § 10]). Kant also regards the "law of the continuum in nature" as a transcendental presupposition of any systematic cognition of nature, but he insists at the same time that the "continuity of forms is a mere idea, for which a corresponding object can by no means be displayed in experience" (CPR B 281-2, 688-9).
23. In *Capital*, famously, Marx also recognizes "the law discovered by Hegel, in his *Logic*, that at a certain point merely quantitative differences pass over by a dialectical inversion into qualitative distinctions" (Marx [1990], 423).
24. See Doz (1970), 170. Doz notes that the first form of quantitative change leads to the emergence of something *else*, whereas the second leads to a new combination of the *same* elements.
25. See Ruschig (1997), 252-5, 262-3, however, for the case of an acid and an alkali that generate a short nodal line. In this case, the line does not embody the category of "nodal line" perfectly, since each new compound begins to form *before* there is a "jump-like" change in the pH-value and so does not suddenly appear with the latter – though the fact that there is a sudden change in *pH-value* would seem to fit well with Hegel's idea of a nodal line.
26. See also PN 2: 434, and Ruschig (1997), 280-4. Ruschig confirms that Hegel is right to state "that oxides do not pass over into one another gradually and that between them no specifically distinct combinations exist" (283; see also 290). Yet he claims – in my view, mistakenly – that Hegel cannot justify the idea of a nodal line by purely logical means, but can derive it only by presupposing chemical examples (291).
27. In nature, of course, it is not just a matter of the quantity or ratio, but the natural conditions also have to be right for the new compound to form.
28. See Burbidge (2006), 57.
29. Acids, bases and gases are conceived here as "measures", rather than just "somethings", because they are characterized not just by certain qualities, but also by certain quantities – the quantities in which they combine with other measures, as well as the quantities that enable to them to be what they are in the first place.
30. See also EL 172 / 227 [§ 109]; Winfield (2012), 146; Martin (2012), 129, and Schick (2014), 141.
31. See this chapter, note 18.
32. See WLS 261.
33. SL 323-4 (l. 38-l.6) / LS 417 (ll. 21-9).

34. See SL 324 / LS 418: "the continuing of the qualitative".
35. Di Giovanni omits the words "of a ratio".
36. See Pechmann (1980), 233: "in the measureless the infinite that is for itself had arisen as the reflected unity of qualities".
37. Di Giovanni translates "Sache" as "fact".
38. See SL 153, 185-6 / LS 193, 235-6.
39. Note that the idea of a substrate is absent from the first edition of the *Logic*, and there is not the same emphasis on the continuing of the qualitative. Hegel emphasises, rather, that quality and quantity pass over into one another and thereby together form an enduring unity (WLS 263-4). This suggests that what remains the same in the nodal line – the truly infinite in the sphere of measure – only comes to be conceived as the *substrate* of change when it is understood to be the *qualitative* continuity that is mediated by, but also indifferent to, quantitative change. – On the differences between the two versions of the logic of measure in the first and second editions of the *Logic*, see also Burbidge (1996), 56-64.
40. On the relation between essence, illusory being and "positing" in Hegel's logic of essence, see Houlgate (2011), 139-44.
41. See Carlson (2007), 231: "The Substrate is a qualitative continuity".
42. This in turn means that measure-combinations now have a new status: those considered in 1.3.2.A.c were understood to be independent and "exclusive", but those we are considering now have been reduced to, or "sublated" into, mere *moments* of a process, their quality being simply an "external state determined by the quantum" (SL 325 / LS 419).
43. See SL 118 / LS 148-9. True quantitative infinity can also be understood as the process of its moments, insofar as it consists in a quantum's "going-beyond-itself into another quantum", that is, in its raising itself to a power of itself. Yet such infinity also consists in the *ratio* between a quantum and its power, in which that quantum relates to itself in another form; indeed, it consists principally in this "ratio of powers", in which the quantum is "for itself". In this respect true quantitative infinity is not as overtly "processual" in character as true qualitative infinity. See SL 278-9 / LS 359-61.
44. See SL 324 / LS 417: "in the measure, the thing [*Sache*]" – that which persists – "is itself already in itself [*an sich*] the unity of the qualitative and the quantitative".
45. The quantum or range of quanta that sustains the quality of each component must, however, endure; otherwise the element itself will be destroyed.
46. The real measure is already a unity before it is conceived as the combination of measures (see this chapter, note 1, and 2: 278, 284-5). As a combination, however, the real measure is a fully *explicit* unity of two ratios.
47. Ruschig states at various points that the substrate is "measureless" (Ruschig [1997], 335, 338; see also Carlson [2007], 234-5, and Schick [2014], 141). This, however, cannot be straightforwardly true. The substrate underlies the various measures that arise in the nodal line, and in that sense might be thought reside "beyond measure". Yet it comprises the qualitative elements that remain constant throughout the line and these have their own measures. Indeed, it is as such that they specify that certain quantitative combinations of them will form new measures in their own right, while others will not. In that sense, the substrate consisting of those elements cannot simply be "measureless".

Chapter 14

1. See AS 2: 46, 56, 60; Doz [1970], 172-3, and LHP 3: 202 / VGP 4: 180. Ruschig points out, however, that whereas, for Hegel, indifference has been "deduced", for Schelling it is "an underived and underivable origin" (Ruschig [2000a], 66, 71). See AS 2: 60: "that which is original" (*das Ursprüngliche*). – Note that di Giovanni translates "Indifferenz" as "indifferentness".
2. Note that LS has "Substanz" (substance), rather than "Substrat". However, the equivalent volume in Hegel's *Gesammelte Werke*, vol. 21, has "Substrat"; see LSGW 373.
3. See SL 326 (l. 14) / LS 420 (l. 12).
4. So, as we move from the first to the second paragraph of 1.3.3.A, indifference does not cease being *ambiguous*; it just proves to be ambiguous in a subtly different way.
5. Hegel does not state explicitly that we move to a new indifference at this point, but the difference between an inverse ratio and a nodal line makes it clear that we do.
6. Di Giovanni translates "das untrennbare Selbständige" as "the one indivisible independent measure".
7. See Doz [1970], 175: "constant sum" (*somme constante*), and Schick [2014], 149: "total quantum" (*Gesamtquantum*).
8. See SL 327 / LS 421: "It is only abstract determinacy that falls in indifference".
9. David Gray Carlson maintains that "we are not to think that the Substrate is therefore the *sum* of these quanta" (Carlson [2007], 242). In doing so, however, he is misled by Hegel's claim that indifference is not a quantum "in itself" (*an ihr selbst*). For Hegel, the substrate is, indeed, the "sum" of the two quanta concerned; yet it is not a sum in itself – purely through being an indifferent substrate – but only "in *relation*" to the quanta it underlies, that is, only as the sum of *those* quanta (SL 327 / LS 421).
10. See Schick [2014], 149.
11. For Hegel's account of the first "formal" inverse ratio, see SL 274 ff. / LS 353 ff.
12. Note that whereas, for Hegel, indifference makes necessary a qualitative as well as a quantitative difference between moments, for Schelling (in his "philosophy of identity") "absolute identity" requires, and permits, there to be "*no other than a quantitative difference*" (AS 2: 55 ff.; see also Doz [1970], 174).
13. See Schick [2014], 149. See also AS 2: 56, where Schelling, too, states that the two moments posited by absolute identity – in his case, subjectivity and objectivity – are posited alternately as "preponderant" (*überwiegend*), even though, in contrast to Hegel, he sees no qualitative difference or "opposition" (*Gegensatz*) between the moments (see AS 2: 55, and this chapter, note 12).
14. But see also 2: 335.
15. See SL 328 (ll. 30-1), 334 (ll. 1-4) / LS 422 (ll. 35-6), 429 (ll. 32-6).
16. See Doz [1970], 176.
17. In the following sentence Hegel makes the same point in a slightly different way when he states that the "more" by which one moment exceeds the other "*would only be the other again*". In LS this statement is difficult to comprehend as it stands. The phrase is, however, derived from a similar one in the first edition of the *Logic*, in which Hegel explains that, in the inverse ratio of indifference, any excess of one

term over the other has to come *from* the other. So in an inverse ratio that is initially 10 : 10, and in which the sum remains 20, one 10 can increase to 11 or 12 only by taking units away from the other. The excess or "more" that distinguishes the greater amount from the other one is thus "*only this other itself*" (WLS 269). Hegel's remark in LS can therefore be read as saying that, if the qualities are strictly inseparable, the excess of one quality over another is in truth *not* an excess, since it in fact belongs *to the other*, and that both qualities are thus present in equal amounts.

18. See AS 2: 59, where Schelling writes that, if we could see all that is "in the totality", we would discern a "complete quantitative equilibrium [*Gleichgewicht*] of subjectivity and objectivity".
19. See Schick [2014], 149-50. Moretto highlights a different contradiction in the current inverse ratio, namely that if the quantum of one quality continues to be increased, the other will disappear altogether (Moretto [2002], 94). Hegel makes this point on SL 330 (ll. 7-11) / LS 424 (ll. 31-5), and then again in the following remark. This contradiction is, of course, at odds with the first contradiction which leads to the conclusion that two truly *inseparable* qualities cannot be present in *different* amounts at all. The two contradictions together, however, constitute what Hegel calls the "all round" contradiction that implicitly besets the inverse ratio of indifference (SL 330 / LS 424).
20. In contrast to the two "sides" of the ratio, X and Y, the "moments" or "factors" to which Hegel refers (on SL 330 / LS 424) are the two qualities, A and B, contained in each "side" in different amounts. In the first edition, by contrast, the sides of the ratio are themselves referred to as "factors". This reflects the fact that in the first edition indifference takes the form of a simple inverse ratio, rather than (as in LS) an inverse ratio of inverse ratios. The two sides of the inverse ratio in the first edition are each the "unity of the qualitative and quantitative", but they are not inverse ratios in themselves (see WLS 267-9).
21. The "whole" Hegel refers to at this point in LS is actually one that results when the quantum of one quality increases to the point at which the other completely disappears (see this chapter, note 19). In my view, however, the logic of indifference that we have been considering also results in such a "whole", so I have made use of the term here. In the first edition Hegel claims that the resulting whole is "not a sum or quantum, nor otherwise a qualitative determinacy" (WLS 269). As I understand it, however, the logic of indifference requires this whole to be both quantitative and qualitative in one. This is implied in LS and also in EL (see EL 172-3 / 228-9 [§§ 110-11]).
22. For Hegel's explicit mention of Brahe and Kepler, see EN 72 / 94 [§270 A], and Houlgate (2005a), 148-51.
23. See this chapter, note 19.
24. Di Giovanni has: "into the predominating strength *of* the centrifugal force" (emphasis added). The German text, however, reads: "in eine überwiegende Stärke *gegen* die Zentrifugalkraft" (emphasis added). Miller gets this right and I have used his translation here (see SLM 381).
25. SL 332 (ll. 3-4) / LS 427 (ll. 8-10).
26. SL 332 (ll. 17-19) / LS 427 (ll. 26-9).

27. Di Giovanni translates “zerstören” (destroy) as “wreck” and “Leerheit der Theorie” (emptiness of the theory) as “trivialization of the theory”.
28. Hegel does not deny that the understanding can conceive of such forces as “independent” of one another, but in this case the “unity of the concept”, and of the matter itself (the motion of planets), would be lost (SL 332 / LS 427). – Note that Hegel criticizes not only the misuse of the final category of measure to comprehend planetary movement, but also the idea that such movement is the product of “*forces*” (see e.g. EN 65 / 85 [§ 269 A]: “We must not therefore speak of forces”, and Houlgate (2005a), 133, 153-6). In the section on quality in the *Logic*, he also criticizes the idea that repulsion and attraction should be understood as forces (see SL 145 / LS 184, and 1: 278). He does, however, introduce “force” into the discussion of crystals in the philosophy of nature (see EN 193 / 239-40 [§ 319 R]).
29. Note that Hegel is here discussing Spinoza’s conception of substance in particular, not substance as it emerges in the doctrine of essence. On the latter, see Houlgate (2000).
30. Spinoza (1994), 85 [*Ethics* ID4], emphasis added.
31. Spinoza (1994), 90 [*Ethics* IP10Schol.].
32. It is, however, true that, for Spinoza, there is no “negation” in substance, since attributes, though different, “involve no negation” (Spinoza (1994), 86 [*Ethics* ID6Exp.]. Attributes, as Spinoza conceives them, differ immediately without negating or limiting one another.
33. Di Giovanni translates “an sich” as “de facto”.
34. See also SL 328-9 / LS 422-3, and 2: 349.

Chapter 15

1. A fairly concentrated, but nonetheless useful, review of the whole logic of measure can be found in Pechmann (1980), 240-68.
2. Di Giovanni translates “Grundlage” as “substrate”.
3. Note that the one (*Eins*) is also, explicitly, “*undifferentiatedness*” (see SL 132 / LS 166, and 1: 259). The differences that are sublated, however, do not belong to the one itself, but precede it. The one is thus simply the explicit *absence* of such differences and so is empty, or “the void”, within itself. In the case of indifference, however, the differences that are sublated are its *own* differences. When indifference undermines itself, therefore, differences are not just absent (as in pure being), or present as absent (as in the one), but they are *present as sublated*. Accordingly, they point back explicitly to the process in which they are sublated (or sublate themselves).
4. See WLS 275: “*being* [. . .] *as this simple negativity of itself*, is *essence*”.
5. In the doctrine of essence, essence itself – paradoxically – first appears as “*affirmative* [*seiendes*], immediate essence”. Yet it is from the start “*sublated being*” – sublated, negated immediacy – and so “being is only a *negative* in relation to essence” (SL 341 / LW 8, emphasis added). Essence is the absence of being or immediacy most explicitly as “reflexion” or “pure negativity” – that is, as the “*movement from nothing to nothing and thereby back to itself*” (SL 346 / LW 14).
6. See also SL 334 / LS 430: “unity *that is for itself*” (fürsichseiende *Einheit*).
7. In the first edition of the *Logic* Hegel states that, as pure self-relation or “pure equality with itself”, essence is itself “simple immediacy”: “The truth of being is

thus to be immediate as absolutely sublated immediacy" (WLS 274). Note that in this respect essence, as it emerges from being, is reminiscent of quantity, as it emerges from quality (see 1: 289-91).

8. See also EL 173 / 229 (§ 111): "simple self-relation that contains within it being as such and its forms as sublated".
9. See Rohs (1982), 57, but also this chapter, note 5. On the changing (and complex) relation between essence and the immediacy of being in the sphere of essence, see Houlgate (2011). – Note, by the way, that in the doctrine of essence (1813) Hegel does not initially mention quality, quantity or measure individually, but he talks only of "being" or "immediacy". In light of what he says at the end of the 1832 doctrine of being, however, we should understand the latter to incorporate – implicitly – the former. So when Hegel states, in the doctrine of essence, that "*being is illusory being*" (*Das Sein ist Schein*), we should understand this to mean that, within the sphere of essence, quality, quantity and measure are all (at first) reduced to mere illusion (SL 341-2 / LW 8-9). Similarly, when external reflexion subsequently "presupposes" being, we should take this to mean that it implicitly presupposes the different determinations of being (SL 348-9 / LW 17-19).
10. The same is true of quantitative infinity which, from the start, coincides explicitly with the finite quantum (see e.g. 2: 160-1). Extensive and intensive magnitude, however, do not include one another explicitly within themselves but turn dialectically into one another (see 2: 141-2, 148-9).
11. See Houlgate (2014), 20. On the close relation between quantity and essence, see also SL 339 / LW 5: "Essence is in the *whole* what *quantity* was in the sphere of being". (Di Giovanni has "quality" instead of "quantity".)
12. See this chapter, note 9.
13. Note that, for Hegel, all quality has its measure (see SL 288 / LS 371), but quality as such nonetheless differs in its logical structure from measure in its various forms. Equally, all quantity – insofar as it belongs to qualitative things and is not just considered in the abstract – either constitutes, or falls within the range of, a measure. Yet particular quanta within such a range are a matter of indifference to the thing and in that respect "mere" quanta (see vol. 2, chapter 9, note 32); and quantity as such, like quality, differs in its logical structure from measure in its various forms.
14. Di Giovanni translates "seiend" as "existent".
15. This does not mean that there will be no further development beyond essence, but there will be no further development, and no further forms, of *being as such*.
16. This account revises the claim I have made elsewhere that essence arises because "there is in truth no simple immediacy in the sphere of being" (see Houlgate [2011], 140).
17. In this sense essence resembles quantity, which is quality in a new form *that is no longer that of quality itself.*
18. See Houlgate (2011), 141, and Martin (2012), 132.
19. See SL 423, 427 / LW 109, 113-14: "existence [*Existenz*] [. . .] is the immediacy of being and the thing is thereby subjected to alteration", and Houlgate (2011), 152-3.
20. See SL 477 ff., 529 ff. / LW 174 ff., LB 32 ff. The concept may thus be said to resemble measure, which is the unity of quality and quantity.
21. See SL 752-3 / LB 305-6, and Houlgate (2005a), 106-10, 171-3.

Chapter 16

1. See 1: 175-7, and 2: 153-4.
2. Bowman (2017), 232. See also vol. 1, chapter 10, note 22.
3. In 1.1.2.B.c.α Hegel points out that the understanding fails to comprehend finitude properly by making it "*absolute*" (SL 102 / LS 127). This remark, however, is peripheral to Hegel's own account of what finitude itself *is* (see 1: 215-17).
4. For Hegel's account of the logical structure of life, see SL 676-88 / LB 211-27.
5. See Houlgate (2005a), 138-44, 147-53.

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